

# O-ASYMPTOTIC CLASSES OF FINITE STRUCTURES

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**Abstract.** In this paper we introduce the concept of  $\mathcal{O}$ -asymptotic classes of finite structures, melding ideas coming from 1-dimensional asymptotic classes and  $\mathcal{O}$ -minimality. The results we present here include a cell-decomposition result for  $\mathcal{O}$ -asymptotic classes and a classification of their infinite ultraproducts: Every infinite ultraproduct of structures in an  $\mathcal{O}$ -asymptotic class is superrosy of  $U^p$ -rank 1, and  $NTP_2$  (in fact, inp-minimal).

## 1. INTRODUCTION

In [17], Macpherson and Steinhorn developed the notion of 1-dimensional asymptotic classes, which are classes of finite structures with notions of measure and dimension coming from the study of the size of their definable sets. Specifically, they define

**Definition 1.1.** *Let  $\mathcal{L}$  be a first order language, and  $\mathcal{C}$  be a collection of finite  $\mathcal{L}$ -structures. Then  $\mathcal{C}$  is a 1-dimensional asymptotic class if the following hold for every  $m \in \mathbb{N}$  and every formula  $\varphi(x, \bar{y})$ , where  $\bar{y} = (y_1, \dots, y_m)$ :*

- (1) *There is a positive constant  $C$  and a finite set  $E \subseteq \mathbb{R}^{>0}$  such that for every  $M \in \mathcal{C}$  and  $\bar{a} \in M^m$ , either  $|\varphi(M, \bar{a})| \leq C$ , or for some  $\mu \in E$ ,*

$$||\varphi(M, \bar{a})| - \mu|M|| \leq C|M|^{1/2}.$$

- (2) *For every  $\mu \in E$ , there is an  $\mathcal{L}$ -formula  $\varphi_\mu(\bar{y})$  such that for all  $M \in \mathcal{C}$ ,  $\varphi_\mu(M^m)$  is precisely the set of  $\bar{a} \in M^m$  with*

$$||\varphi(M, \bar{a})| - \mu|M|| \leq C|M|^{1/2}.$$

The seminal example of these classes is the class of finite fields, for which the conditions above appear as a remarkable theorem of Chatzidakis, Macintyre and Van den Dries (see [2]). With this definition, which is a condition on the definable sets in only one variable, Macpherson and Steinhorn could obtain results about the control of the size of definable sets in many variables as well as results concerning the behavior of the infinite ultraproducts of structures in such classes. For instance, they show that if every ultraproduct of the class  $\mathcal{C}$  is strongly minimal, then  $\mathcal{C}$  is a 1-dimensional asymptotic class. Furthermore, they prove that every ultraproduct of a 1-dimensional class is supersimple of  $U$ -rank 1.

An easy example of a class of finite structures which is not a 1-dimensional class is the class of all finite totally ordered sets, which fails property (1) because the formula  $x < a$  can pick out an arbitrary proper initial segment of a structure as  $a$  varies. However, the only definable sets in one variable on the structures of this class (and in their ultraproducts) are finite unions of intervals and points, implying that the structures involved are  $\mathcal{O}$ -minimal.

$\mathcal{O}$ -minimality and its variants are properties that give a good structure theories for infinite ordered structures. Our aim here is to isolate conditions on classes of finite linearly ordered structures to get nice asymptotic properties, melding ideas of asymptotic classes and  $\mathcal{O}$ -minimality.

With this idea in mind, we propose a definition of  $\mathcal{O}$ -asymptotic classes as an adaptation of the definition of 1-dimensional asymptotic classes in the context of totally ordered structures.

This paper is organized as follows: in Section 2 we present the definition of  $\mathcal{O}$ -asymptotic classes, as well as a result about the behavior of the definable sets of these classes in several variables. This result can be seen as a finite combinatorial cell decomposition theorem for ordered structures.

In section 3 we present the main examples of  $\mathcal{O}$ -asymptotic classes. The main example that we will present corresponds to the class of cyclic groups  $(\mathbb{Z}/(2N+1)\mathbb{Z}, +)$  with a natural order induced on the classes  $-\overline{N} < \dots < -\overline{1} < \overline{0} < \overline{1} < \dots < \overline{N}$ .

In section 4 we start the study of the infinite ultraproducts of structures in  $\mathcal{O}$ -asymptotic classes, for which our main interest is to place them in the map of classification theory. Among the main results are: if every ultraproduct of a class of finite totally ordered structures is  $\mathcal{O}$ -minimal, then the class is  $\mathcal{O}$ -asymptotic. Furthermore, every ultraproduct of an  $\mathcal{O}$ -asymptotic class is  $\text{NTP}_2$  and super rosy of  $\text{U}^{\text{p}}$ -rank 1.

The last section is an appendix in which I included some results on uniform quantifier elimination necessary for the examples

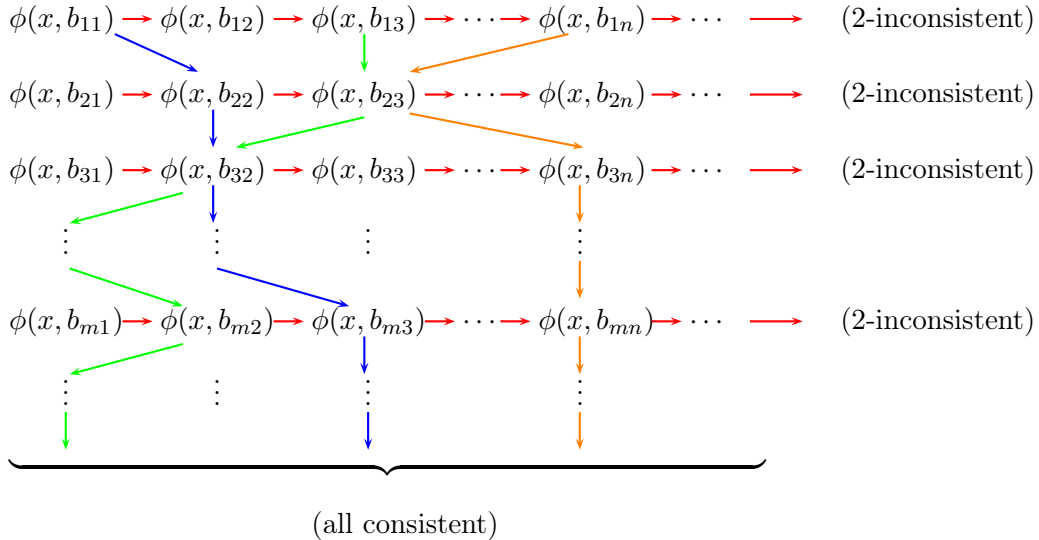
**Acknowledgements:** I am deeply grateful with my advisors Alf Onshuus and Thomas Scanlon for their valuable comments on the development of this research and the corrections made on previous versions of this paper.

**1.1. Rosy theories and theories with  $\text{NTP}_2$ .** During the last years, the class of  $\text{NTP}_2$  theories (that contains both simple theories and theories with NIP) have received attention of the model-theorists, especially after the results in [4, 5, 6]. Even if it was defined in [19, 20], its systematic study was initiated in [4]. Now we give the definitions of these theories.

**Definition 1.2.** A formula  $\phi(x, y)$  has  $\text{TP}_2$  (standing for tree property of the second kind) relative to  $M$  if there is an array  $\langle b_{\alpha, i} : \alpha, i < \omega \rangle$  of elements in  $M$  such that:

- For every  $\alpha < \omega$ , the set  $\{\phi(x, b_{\alpha, i}) : i < \omega\}$  is 2-inconsistent.
- For any function  $f : \omega \rightarrow \omega$ , the set  $\{\phi(x, b_{\alpha, f(\alpha)}) : \alpha < \omega\}$  is consistent.

A way to understand  $\text{TP}_2$  is to think about an “infinite rectangular array” of formulas for which every row is 2-inconsistent, but every “descending path” is consistent.



**Definition 1.3.** We say that  $M$  has  $\text{NTP}_2$  if there is no  $\mathcal{L}$ -formula witnessing the  $\text{TP}_2$  property in  $M$ . A theory  $T$  has  $\text{NTP}_2$  if every model of  $T$  has  $\text{NTP}_2$ .

There are recent results providing examples of natural theories which have  $\text{NTP}_2$  but are not simple theories nor theories with NIP: they include ultraproducts of  $p$ -adics (see [5]) and the

theory of the non-standard Frobenius automorphism acting on an algebraically closed valued field of equicharacteristic 0 (see [6]).

The following results about theories with  $\text{NTP}_2$  will be used in Section 4.

**Fact 1.4.** *If  $T$  has  $\text{TP}_2$ , there is some formula  $\phi(x, \bar{y})$  with  $|x| = 1$  that has  $\text{TP}_2$ .*

**Fact 1.5** (Chernikov, Lemma 2.2 in [4]). *If there is an array  $\langle \bar{b}_{\alpha, i} : \alpha, i < \omega \rangle$  witnessing  $\text{TP}_2$  for the formula  $\phi(x; \bar{y})$  then the rows may be assumed to be mutually indiscernible sequences.*

**1.2.  $\mathfrak{p}$ -Forking and Rosy Theories.** The notion of  $\mathfrak{p}$ -forking (which should be read as *thorn-forking*) was introduced in Alf Onshuus Ph.D. Thesis and appeared as a generalization of forking to contexts in which non-forking failed to provide a nice independence relation (such as  $\mathcal{O}$ -minimal theories).

- Definition 1.6.**
- (1) *A formula  $\phi(\bar{x}, \bar{b})$  strongly divides over  $A$  if  $\bar{b}$  is not algebraic over  $A$  and there is  $k < \omega$  such that the set  $\{\phi(\bar{x}, \bar{b}') : \bar{b}' \models \text{tp}(\bar{b}/A)\}$  is  $k$ -inconsistent.*
  - (2) *We say that  $\phi(\bar{x}, \bar{b})$   $\mathfrak{p}$ -divides over  $A$  if there is a finite tuple  $\bar{c}$  such that  $\phi(\bar{x}, \bar{b})$  strongly divides over  $A\bar{c}$ .*
  - (3) *We say that  $\phi(\bar{x}, \bar{b})$   $\mathfrak{p}$ -forks over  $A$  if there are formulas  $\psi_1(\bar{x}, \bar{c}_1), \dots, \psi_n(\bar{x}, \bar{c}_n)$  such that each  $\psi_i(\bar{x}, \bar{c}_i)$   $\mathfrak{p}$ -divides over  $A$  and  $\phi(\bar{x}, \bar{b}) \vdash \bigvee_{i=1}^n \psi_i(\bar{x}, \bar{c}_i)$ .*
  - (4) *A type  $\mathfrak{p}$ -forks ( $\mathfrak{p}$ -divides) over  $A$  if it implies a formula which  $\mathfrak{p}$ -forks ( $\mathfrak{p}$ -divides) over  $A$ .*
  - (5) *We write  $\bar{a} \downarrow_A^{\mathfrak{p}} B$  (read as  $\bar{a}$  is thorn-independent of  $B$  over  $A$ ) to denote that  $\text{tp}(\bar{a}/B)$  does not  $\mathfrak{p}$ -fork over  $A$ .*

Just like simple theories are those theories on which forking-independence has nice properties and can be characterized as those theories where the forking-independence is symmetric (meaning that  $\bar{a} \downarrow_A \bar{b}$  if and only if  $\bar{b} \downarrow_A \bar{a}$ ), there is a class of theories called *rosy* on which the  $\mathfrak{p}$ -forking has desirable properties, and can be also characterized as those theories where the  $\mathfrak{p}$ -independence is a symmetric independence relation.

- Definition 1.7.**
- (1) *We say that  $q \in S(B)$  is a  $\mathfrak{p}$ -forking extension of  $p \in S(A)$  (with  $A \subseteq B$ ) if  $q$  is an extension of  $p$  and the type  $q$   $\mathfrak{p}$ -forks over  $A$ . Otherwise, we called it a non- $\mathfrak{p}$ -forking extension of  $p$ .*
  - (2) *We define the  $U^{\mathfrak{p}}$ -rank (read as  $U$ -thorn-rank) to be the foundation rank for  $\mathfrak{p}$ -forking. Namely,  $U^{\mathfrak{p}}(p(\bar{x})) \leq 0$  if and only if  $p(\bar{x})$  is consistent,  $U^{\mathfrak{p}}(p(\bar{x})) \geq \alpha + 1$  if and only if there is a  $\mathfrak{p}$ -forking extension  $q(\bar{x})$  of  $p(\bar{x})$  such that  $U(q(\bar{x})) \geq \alpha$  and for a limit ordinal  $\lambda$ ,  $U^{\mathfrak{p}}(p(\bar{x})) \geq \lambda$  if and only if  $U^{\mathfrak{p}}(p(\bar{x})) \geq \alpha$  for every  $\alpha < \lambda$ .*

**Definition 1.8.** *A structure  $M$  is said to be superrosy of  $U^{\mathfrak{p}}$ -rank  $n$  if there is an 1-type  $p(x)$  such that  $U^{\mathfrak{p}}(p(x)) = n$ , but there is no 1-type  $q(x)$  with  $U^{\mathfrak{p}}(q(x)) \geq n + 1$ .*

The basic properties of  $\mathfrak{p}$ -forking and superrosy theories were investigated in [18]. It was shown in that paper that in  $\mathcal{O}$ -minimal theories the notion of  $\mathfrak{p}$ -independence coincides with the usual notion of independence, and the  $U^{\mathfrak{p}}$ -rank corresponds to the  $\mathcal{O}$ -minimal dimension on definable sets.

The class of rosy theories is the class of theories for which  $\mathfrak{p}$ -forking has a nice behavior (for instance, a theory is rosy if and only if the  $\mathfrak{p}$ -independence satisfies symmetry). Also, according to Definition 1.8 we might say that a theory is superrosy of  $U^{\mathfrak{p}}$ -rank  $n$  if  $n$  is the length of the maximal  $\mathfrak{p}$ -dividing chain for 1-types in  $M$ .

It is known that in the presence of a definable order, forking is very different from  $\mathfrak{p}$ -forking. For example, in the theory  $Th(\mathbb{Q}, <)$  we have that the formula  $\varphi(x) := a < x < b$  divides over the empty set (despite the fact that the  $\mathcal{O}$ -minimal dimension of  $\varphi(M)$  is 1), but it does not  $\mathfrak{p}$ -fork over the empty set.

**1.3. A lemma about measure theory.** The following fact is a consequence of the truncated inclusion-exclusion principle, and will be used in Section 4. A proof of it can be found in [9] or [10].

**Fact 1.9.** [10, Proposition 2.2.10] *Let  $X$  be a measure space with  $\mu(X) = 1$  and fix  $0 < \epsilon \leq \frac{1}{2}$ . Let  $\langle A_i : i < \omega \rangle$  be a sequence of measurable subsets of  $X$  such that  $\mu(A_i) \geq \epsilon$  for every  $i$ .*

*Then, for every  $k < \omega$  there are  $i_1 < i_2 < \dots < i_k$  such that*

$$\mu \left( \bigcap_{j=1}^k A_{i_j} \right) \geq \epsilon^{3^{k-1}}.$$

## 2. $\mathcal{O}$ -ASYMPTOTIC CLASSES AND CELL DECOMPOSITION RESULTS

**Definition 2.1.** *Let  $\mathcal{C}$  be a class of finite linearly ordered structures in a language  $\mathcal{L}$  containing  $<$ . We say  $\mathcal{C}$  is a **weak- $\mathcal{O}$ -asymptotic** class if for every  $m \in \mathbb{N}$  and formula  $\varphi(x; y_1, \dots, y_m)$  there is a constant  $C > 0$  and  $k \geq 1$  and a finite set  $E \subseteq ([0, 1])^k$  such that:*

- (1) *For every  $M \in \mathcal{C}$  and  $\bar{a} \in M^m$  there are elements*

$$c_0 = \min M < c_1 < \dots < c_k = \max M$$

*and a tuple  $\bar{\mu} \in E$  such that:*

- (\*) *For every  $i = 1, 2, \dots, k$ , either*

$$\begin{cases} \mu_i = 0 & \text{and } |\varphi(M, \bar{a}) \cap (c_{i-1}, c_i)| \leq C \\ \text{or} \\ \mu_i > 0 & \text{and } ||\varphi(M, \bar{a}) \cap (c_{i-1}, c_i)| - \mu_i|(c_{i-1}, c_i)|| \leq C|(c_{i-1}, c_i)|^{1/2} \end{cases}$$

- (2) *For every  $\bar{\mu} \in E$  there is a formula  $\varphi_{\bar{\mu}}(\bar{y}; z_1, \dots, z_k)$  such that*

$$M \models \varphi_{\bar{\mu}}(\bar{a}; c_1, \dots, c_k) \quad \text{implies} \quad (*) \text{ holds.}$$

**Definition 2.2.** *Let  $\mathcal{C}$  be a class of finite linearly ordered structures in a language  $\mathcal{L}$  containing  $<$ . We say  $\mathcal{C}$  is an  **$\mathcal{O}$ -asymptotic** class if for every  $m \in \mathbb{N}$  and formula  $\varphi(x; y_1, \dots, y_m)$  there is a constant  $C > 0$  and  $k \geq 1$  and a finite set  $E \subseteq ([0, 1])^k$  such that:*

- (1) *For every  $M \in \mathcal{C}$  and  $\bar{a} \in M^m$  there are elements*

$$c_0 = \min M < c_1 < \dots < c_k = \max M$$

*and a tuple  $\bar{\mu} \in E$  such that:*

- (\*) *For every  $i = 1, 2, \dots, k$ , either*

$$\begin{cases} \mu_i = 0 & \text{and } |\varphi(M, \bar{a}) \cap (c_{i-1}, c_i)| \leq C \\ \text{or} \\ \mu_i > 0 & \text{and for every } (u, v) \subseteq (c_{i-1}, c_i), \\ & ||\varphi(M, \bar{a}) \cap (u, v)| - \mu_i|(u, v)|| \leq C|(u, v)|^{1/2} \end{cases}$$

- (2) *For every  $\bar{\mu} \in E$  there is a formula  $\varphi_{\bar{\mu}}(\bar{y}; z_1, \dots, z_k)$  such that*

$$M \models \varphi_{\bar{\mu}}(\bar{a}; c_1, \dots, c_k) \quad \text{implies} \quad (*) \text{ holds.}$$

**Remark 2.3.** Roughly speaking, a class of finite ordered structures is weakly  $\mathcal{O}$ -asymptotic if every formula in one variable admits a decomposition into a fixed number of intervals such that on each interval it behaves like in 1-dimensional classes.  $\mathcal{O}$ -asymptoticity requires also this decomposition to be *uniform* in the sense that the definable set is uniformly distributed along each of the intervals  $(c_{i-1}, c_i)$ . I believe that these two notions are equivalent, but I have not been able to prove this equivalence. The main difficulty here has been the non-additive nature of the “error term”.

Throughout this paper, we will be working with the definition of  $\mathcal{O}$ -asymptotic classes of finite structures.

**Notation:** Assume we are working in an  $\mathcal{O}$ -asymptotic class.

- We say that  $\varphi(x; \bar{a})$  *admits a decomposition with proportion  $\bar{\mu}$*  to mean that there are  $c_1, \dots, c_k$  such that condition (1) of Definition 2.2 holds for  $\varphi(x; \bar{a})$  and  $c_1 < \dots < c_k$ .
- When we say *by uniformity of  $\mu_i$  in  $(c, d)$*  or *by the uniformity of distribution of  $\varphi(x; \bar{a})$  in  $(c, d)$*  we mean that, for every  $(u, v) \subseteq (c, d)$ ,

$$||\varphi(M, \bar{a}) \cap (u, v)| - \mu_i(u, v)|| \leq C|(u, v)|^{1/2}$$

We will implicitly use either  $\mu_i$  or  $\varphi(x; \bar{a})$ , but in any case it will be clear from the context.

**Example 2.4.** The class  $\mathcal{C}_{ord}$  of finite linear orders is an  $\mathcal{O}$ -asymptotic class.

It is easy to show that this class admits uniform quantifier elimination (see Appendix 7.1). Moreover, every formula  $\varphi(x, \bar{a})$  in one variable is a disjunction of formulas of the form

$$\bigwedge_i t_i(\bar{a}) < x \wedge \bigwedge_j x < u_j(\bar{a}) \wedge \bigwedge_l x = v_l(\bar{a})$$

where  $t_i, u_j, v_k$  are terms depending on the tuple  $\bar{a}$  in the language  $\mathcal{L}' = \{\min M, \max M, S, S^{-1}, <\}$ , with  $S$  and  $S^{-1}$  are the successor and predecessor functions respectively. This formula defines a finite union of intervals and points with a bound in the number of intervals. If  $\varphi(x, \bar{a})$  defines at most  $k$  intervals, we may take  $\langle c_i : i \leq k \rangle$  to be the end points of the intervals, and as measures all the possible vectors  $\bar{\mu} \in \{0, 1\}^k$ .

Now we present a definition of  $\mathcal{C}$ -cells, that is analogous to the concept of cells in  $\mathcal{O}$ -minimal theories. The main differences arise from the lack of continuity, the fact that we are dealing with discrete orders that are approximable by finite structures, and the intersection with  $\bar{\mu}$ -definitions. All these reasons make difficult to apply the usual definition of cells in dense or discrete  $\mathcal{O}$ -minimal theories.

**Definition 2.5.** Let  $\mathcal{C}$  be an  $\mathcal{O}$ -asymptotic class and let  $(M, <, \dots)$  be a structure in  $\mathcal{C}$ .

- (1) We define the  $\mathcal{C}$ -cells inductively:
  - A (1) –  $\mathcal{C}$ -cell is a non-empty set of the form  $(a, b)$ .
  - If  $X$  is a  $(i_1, \dots, i_m)$  –  $\mathcal{C}$ -cell and  $\varphi_{\bar{\mu}}(\bar{y}; \bar{z})$  with  $|\bar{y}| = m$  is the  $\bar{\mu}$ -definition for some formula  $\varphi(x; \bar{y})$ , then for every  $\bar{d}$  with  $|\bar{d}| = |\bar{z}|$  we have that if the set  $X \cap \varphi_{\bar{\mu}}(\bar{y}; \bar{d})$  is non-empty, it is also a  $(i_1, \dots, i_m)$  –  $\mathcal{C}$ -cell.
  - Let  $X$  be a  $(i_1, i_2, \dots, i_n)$  –  $\mathcal{C}$ -cell.
    - If  $f : M^n \rightarrow M$  is a definable function, then  $\Gamma(f) = \{(\bar{x}, f(\bar{x})) : \bar{x} \in X\}$  is a  $(i_1, \dots, i_m, 0)$  –  $\mathcal{C}$ -cell.
    - If  $f, g : M^n \rightarrow M$  are definable functions with  $f|_X < g|_X$ , then the set  $(f, g)_X = \{(\bar{x}, y) : f(\bar{x}) < y < g(\bar{x}) : \bar{x} \in X\}$  is a  $(i_1, \dots, i_m, 1)$  –  $\mathcal{C}$ -cell.
- (2) A  $\mathcal{C}$ -cell decomposition for  $M^n$  is a partition  $\{Z_1, \dots, Z_k\}$  of  $M^n$  such that every  $Z_i$  is a  $\mathcal{C}$ -cell.

(3) If  $Z$  is a  $\mathcal{C}$ -cell in  $M^k$ , we can define  $\dim(Z)$  as

$$\dim(Z) = \min \left\{ \sum_{j=1}^k i_j : \text{the cell } Z \text{ can be expressed as a } (i_1, \dots, i_k) - \mathcal{C}\text{-cell} \right\}$$

**Definition 2.6.** We say that  $Z \subseteq M^k$  is an 1-cell if there is some  $j \leq k$  such that the projection of  $Z$  under the  $j$ -th coordinate is a bijection between  $Z$  and a (1)-cell in  $M^1$ .

**Remark 2.7.**

- Note that by definition, for a  $(i_1, \dots, i_k) - \mathcal{C}$ -cell it is always true that  $i_1 = 1$ . Also, it is possible that a (1)-cell is a point, and this will happen when  $S^2(a) = b$  (with  $S$  the successor function).
- Moreover, a  $(1, 0) - \mathcal{C}$ -cell might also be described as a  $(1, 1) - \mathcal{C}$ -cell. For instance,  $\{x \in (a, b) : f(x) = c\} = \{x \in (a, b) : S^{-1}c < f(x) < Sc\}$ .
- The concept of 1-cell (without parenthesis) should not be confused with (1)-cells. In general, 1-cells can be subsets of  $M^k$  for any  $k$ , while (1)-cells are always intervals in  $M^1$ .

Note that the cells in this context are allowed to be the intersection of  $\mathcal{O}$ -minimal cells (the ones produced by intervals and definable functions) with some other definable sets (the ones given by the definition for some formula  $\varphi$  and measure  $\bar{\mu} \in E_\varphi$ ). So, in some sense, these cells look more like the cells in Presburger Arithmetic used by Cluckers in [7] to get the cell-decomposition theorem for  $(\mathbb{Z}, +, <)$  than the usual cells used in  $\mathcal{O}$ -minimal theories.

Given a formula  $\varphi(x, \bar{y})$  and  $\bar{a} \in M$  we have that, once the measure  $\bar{\mu} \in E$  for  $\varphi(x, \bar{a})$  is known, the condition (2) of being an  $\mathcal{O}$ -asymptotic class provides a canonical way to choose the decomposition into intervals for  $\varphi(M; \bar{a})$ :

Consider the definable functions  $m_1^\varphi, \dots, m_k^\varphi$  with parameters  $\bar{a}$ , defined as follows:

- $M \models m_1^\varphi(\bar{a}) = w$  if and only if
 
$$M \models \exists w_2, \dots, w_k (\varphi_{\bar{\mu}}(\bar{a}, w, w_2, \dots, w_k))$$

$$\wedge \forall v (v < w \rightarrow \neg \exists w_2, \dots, w_k (\varphi_{\bar{\mu}}(\bar{a}, w, w_2, \dots, w_k)))$$
- $M \models m_2^\varphi(\bar{a}) = w$  if and only if
 
$$M \models \exists w_3, \dots, w_k (\varphi_{\bar{\mu}}(\bar{a}, m_1^\varphi(\bar{a}), w, \dots, w_k))$$

$$\wedge \forall v (v < w \rightarrow \neg \exists w_3, \dots, w_k (\varphi_{\bar{\mu}}(\bar{a}, m_1^\varphi(\bar{a}), w, w_3, \dots, w_k)))$$
- $M \models m_k^\varphi(\bar{a}) = w$  if and only if
 
$$M \models \varphi_{\bar{\mu}}(\bar{a}, m_1^\varphi(\bar{a}), \dots, m_{k-1}^\varphi(\bar{a}), w)$$

$$\wedge \forall v (m_{k-1}^\varphi(\bar{a}) < v < w \rightarrow \neg (\varphi_{\bar{\mu}}(\bar{a}, m_1^\varphi(\bar{a}), m_2^\varphi(\bar{a}), \dots, m_{k-1}^\varphi(\bar{a}), v)))$$

Basically,  $m_1^\varphi(\bar{a}) = w$  if and only if  $w$  is the first element that can be used in a “good decomposition” for  $\varphi(x, \bar{a})$ , and we may define the functions  $m_i^\varphi(\bar{a})$  as the  $i$ -th element in the “minimal” decomposition for  $\varphi(x, \bar{a})$  ( $i = 2, \dots, k$ ).

These functions will play a key role in the proof of the following result, which can be seen as an analogue of Theorem 2.1 of [17], combined with ideas from cell decomposition in  $\mathcal{O}$ -minimal theories.

**Theorem 2.8.** Suppose  $\mathcal{C}$  is an  $\mathcal{O}$ -asymptotic class of finite  $\mathcal{L}$ -structures. Then for every  $\mathcal{L}$ -formula  $\varphi(\bar{x}, \bar{y})$  (where  $|\bar{x}| = n, |\bar{y}| = m$ ) the following hold:

- (1) *There is a positive constant  $C > 0$ ,  $k = k(\varphi, n) \in \mathbb{N}$  and a finite set  $D$  of tuples  $\bar{\alpha}$  with  $\bar{\alpha} \in [0, 1]^k$  such that for every  $M \in \mathcal{C}$  and  $\bar{a} \in M^m$ , there is a cell decomposition  $\{Z_1, \dots, Z_k\}$  of  $M^n$  into  $k$  cells such that:*  
 (\*) *For every  $i \leq k$ , either*

$$\begin{cases} \alpha_i = 0 \text{ and } |\varphi(M^n; \bar{a}) \cap Z_i| \leq C \\ \text{or} \\ \alpha_i \neq 0 \text{ and } |\varphi(M^n; \bar{a}) \cap Z_i| - \alpha_i |Z_i| \leq C |L_i|^{\dim(Z_i)-1/2} \end{cases}$$

where  $L_i$  is a 1-cell of maximal size contained in  $Z_i$ .

- (2) *For every  $\bar{\alpha} \in D$  there are  $\mathcal{L}$ -formulas  $\varphi_{\bar{\alpha}}(\bar{y}; \bar{z})$  and  $Z_1(\bar{x}; \bar{z}), \dots, Z_k(\bar{x}; \bar{z})$  such that  $M \models \psi_{(\bar{\alpha})}(\bar{a}; \bar{d})$  if and only if*  
 •  $\{Z_1(\bar{x}; \bar{d}), \dots, Z_k(\bar{x}; \bar{d})\}$  *is a cell decomposition of  $M^n$ .*  
 • (\*) *holds for this decomposition.*

*Proof.* We will prove this by induction on  $n$ .

•  **$n = 1$  case:**

Let  $\varphi(x; \bar{y})$  be an  $\mathcal{L}$ -formula. By the definition of  $\mathcal{O}$ -asymptotic classes, there is a constant  $C > 0$ ,  $k \geq 1$  and  $E \subseteq^{fin} [0, 1]^k$  such that:

- (i) For all  $M \in \mathcal{C}$  and  $\bar{a} \in M^m$  there are  $c_0 = \min M < c_1 < \dots < c_k = \max M$  and  $\bar{\mu} \in E$  such that for every  $i = 1, 2, \dots, k$  either

$$||\varphi(M, \bar{a}) \cap (c_{i-1}, c_i)| - \mu_i(c_{i-1}, c_i)|| \leq C |(c_{i-1}, c_i)|^{1/2} \quad (*)$$

or  $|\varphi(M, \bar{a}) \cap (c_{i-1}, c_i)| \leq C$ . Here we can take  $Z_{2i} = c_i$  and  $Z_{2i+1} = (c_{i-1}, c_i)$  for  $i = 0, \dots, k$  that clearly is a cell-decomposition for  $M$ . Also, we can take  $D$  to be the finite set of measures  $\bar{\alpha} \in [0, 1]^{2k}$  given by  $\alpha_{2i} = 0, \alpha_{2i+1} = \mu_i$ , for  $\bar{\mu} \in E$ .

- (ii) For every  $\bar{\mu} \in E$  there is an  $\mathcal{L}$ -formula  $\psi_{\bar{\mu}}(\bar{y}; z_1, \dots, z_{k-1})$  such that  $M \models \psi_{\bar{\mu}}(\bar{a}; c_1, \dots, c_{k-1})$  if and only if (\*) holds. We can take now the formula

$$\varphi_{\bar{\mu}}(\bar{a}; c_1, \dots, c_{k-1}) = \psi_{\bar{\mu}}(\bar{a}; c_1, \dots, c_{k-1}) \wedge (\min M < c_1 < \dots < c_{k-1} < \max M).$$

• **Inductive case:**

Let  $\varphi(z, \bar{x}; \bar{y})$  be an  $\mathcal{L}$ -formula with  $1 + n + m$  variables, with  $|\bar{x}| = n, |\bar{y}| = m$ . The measures for this formula will be denoted by  $\alpha$ , so when using the case  $n = 1$  or the inductive hypothesis, we will be using different letters.

Considering  $(\bar{x}; \bar{y})$  as parameter variables, we have by the  $n = 1$  case that for every  $(\bar{b}, \bar{a}) \in M^n \times M^m$ , there are  $c_0 = \min M < c_1 < \dots < c_k = \max M$  in  $M$  and an element  $\bar{\mu}_{\bar{b}} = (\mu^1, \dots, \mu^k) \in E$  such that:

- (\*) For every  $i = 1, 2, \dots, k$ , either

$$\begin{cases} \mu_i = 0 \text{ and } |\varphi(M, \bar{b}; \bar{a}) \cap (c_{i-1}, c_i)| \leq C \\ \text{or} \\ \mu_i > 0 \text{ and } ||\varphi(M, \bar{b}; \bar{a}) \cap (c_{i-1}, c_i)| - \mu_i(c_{i-1}, c_i)|| \leq C |(c_{i-1}, c_i)|^{1/2}. \end{cases}$$

Furthermore, there is a formula  $\varphi_{\bar{\mu}_{\bar{b}}}(\bar{x}, \bar{y}, w_1, \dots, w_k)$  such that  $M \models \varphi_{\bar{\mu}_{\bar{b}}}(\bar{b}, \bar{a}, c_1, \dots, c_k)$  if and only if (\*) holds.

Note that the formulas above depend only on the associated measure  $\bar{\mu} \in E$ . So, we can enumerate  $E$  by  $E = \{\bar{\mu}_1, \dots, \bar{\mu}_t\}$  and  $\varphi_1, \dots, \varphi_t$  the corresponding formulas. ( $\varphi_i := \varphi_{\bar{\mu}_i}$ ). Consider the formulas

$$\phi_i(\bar{x}; \bar{y}) := \exists z_1, \dots, z_k (\varphi_i(\bar{x}, \bar{y}; z_1, \dots, z_k)).$$

Note that given  $\bar{a} \in M^m$ , the formulas  $\phi_1(\bar{x}; \bar{a}), \dots, \phi_t(\bar{x}; \bar{a})$  form a partition of  $M^n$ . Also, by induction hypothesis, for each formula  $\phi_i(\bar{x}; \bar{y})$  there is a finite set  $D_i = \{\bar{v}_1^i, \dots, \bar{v}_u^i\} \subseteq [0, 1]^l$  and formulas  $Z_{i,1}(\bar{x}, \bar{y}; \bar{w}), \dots, Z_{i,\ell}(\bar{x}, \bar{y}; \bar{w})$  such that:

- For every  $\bar{a} \in M^m$  there are  $\bar{v} \in D_i$  and a tuple  $\bar{d}$  satisfying:
- $\{Z_{i,1}(\bar{x}; \bar{a}, \bar{d}), \dots, Z_{i,\ell}(\bar{x}; \bar{a}, \bar{d})\}$  is a cell decomposition of  $M^n$ .
- $(*)_{i,n}$ : For every  $j \leq \ell$ , either

$$\begin{cases} \nu_j = 0 & \text{and } |\varphi_i(M^n; \bar{a}) \cap Z_{i,j}(M^n; \bar{d})| \leq C \\ \text{or} \\ \nu_j \neq 0 & \text{and } ||\varphi_i(M^n; \bar{a}) \cap Z_{i,j}(M^n; \bar{d})| - \nu_j|Z_{i,j}(M^n; \bar{d})| \leq C|Z_{i,j}(M^n; \bar{d})|^{1/2}. \end{cases}$$

Furthermore, for every  $\bar{v} \in D_i$  there is a formula  $\chi_{\bar{v}}^i(\bar{y}; \bar{z})$  such that  $M \models \chi_{\bar{v}}^i(\bar{a}; \bar{d})$  if and only if the conditions above hold. As we did before, we can enumerate these formulas and let  $\chi_h^i(\bar{y}; \bar{z})$  be the formula corresponding to the measure  $\bar{v}_h \in D_i$  ( $h \leq u$ ).

Note that the functions  $m_i^\varphi(\bar{y})$  defined previously have higher-dimensional generalizations for  $n$ -variable formulas, obtained after using the induction hypothesis and the lexicographical order. We will call these functions as  $m_i^{\varphi,n}(\bar{y})$ . The index  $n$  could be implicit if the formula  $\varphi(\bar{x}; \bar{y})$  satisfies  $|x| = n$ , but we will omit sometimes the formula and for clarity will always put the index  $n$  when necessary. So, for instance, the formulas  $Z_{i,j}(\bar{x}; \bar{a}, \bar{d})$  providing the cell-decomposition of  $M^n$  for  $\phi_i$  can be written as  $Z_{i,j}(\bar{x}; \bar{a}, m^n(\bar{a}))$ .

Consider now the following sets of formulas:

$$\begin{aligned} \mathfrak{X}(z, \bar{x}; \bar{y}) &:= \{Z_{i,j}(\bar{x}; m^n(\bar{y})) \wedge \varphi_i(\bar{x}; \bar{y}, m_1(\bar{x}, \bar{y}), \dots, m_k(\bar{x}, \bar{y})) \wedge (m_s(\bar{x}, \bar{y}) < z < m_{s+1}(\bar{x}, \bar{y}))\} \\ &\cup \{Z_{i,j}(\bar{x}; m^n(\bar{y})) \wedge \varphi_i(\bar{x}; \bar{y}, m_1(\bar{x}, \bar{y}), \dots, m_k(\bar{x}, \bar{y})) \wedge (m_s(\bar{x}, \bar{y}) = z)\} \end{aligned}$$

for  $i \leq t, j \leq u, 0 \leq s \leq k$ .

It is clear that for every  $\bar{a} \in M^m$ , the collection  $\mathfrak{X}(z, \bar{x}; \bar{a})$  forms a  $\mathcal{C}$ -cell decomposition of  $M^{n+1}$ , because the sets defined by

$$Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}, \bar{y}, m_1(\bar{x}), \dots, m_k(\bar{x}, \bar{y}))$$

are  $\mathcal{C}$ -cells,  $m_i(\bar{x})$  are definable functions, and originally the formulas  $Z_{i,j}$  and  $\varphi_i$  formed a partition of the space  $M^n$ .

Consider now the following refinement of  $\mathfrak{X}(z, \bar{x}; \bar{a})$ :

- (a) If  $\mu_i^s = 0$ , then for every  $\bar{b} \models Z_{i,j}(\bar{x}; m^n(\bar{a})) \cap \phi_i(\bar{x}; \bar{a})$  we have

$$|\{z : (m_s(\bar{b}, \bar{a}) < z < m_{s+1}(\bar{b}, \bar{a})) \wedge \varphi(z, \bar{b}; \bar{a})\}| \leq C$$

and we can take the following definable functions:

$$\begin{aligned} f_{1,s}(\bar{x}) &= \min \{z : m_s(\bar{x}) < z < m_{s+1}(\bar{x}) \wedge \varphi(z, \bar{x}; \bar{a})\} \\ f_{2,s}(\bar{x}) &= \min \{z : f_{1,s}(\bar{x}) < z < m_{s+1}(\bar{x}) \wedge \varphi(z, \bar{x}; \bar{a})\} \\ &\vdots \\ f_{C,s}(\bar{x}) &= \min \{z : f_{C-1,s}(\bar{x}) < z < m_{s+1}(\bar{x}) \wedge \varphi(z, \bar{x}; \bar{a})\} \\ &= \max \{z : z < m_{s+1}(\bar{x}) \wedge \varphi(z, \bar{x}; \bar{a})\}. \end{aligned}$$

Thus, we have a refinement of the original  $\mathcal{C}$ -cell given by the collection of  $\mathcal{C}$ -cells

$$\begin{aligned} &\{Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{y}), \dots, m_k(\bar{x}, \bar{y})) \\ &\wedge (f_{r,s}(\bar{x}) < z < f_{r+1,s}(\bar{x})) : 0 \leq r \leq C+1\} \\ &\cup \{Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{y}), \dots, m_k(\bar{x}, \bar{y})) \wedge (f_{r,s}(\bar{x}) = z) : 0 \leq r \leq C+1\}, \end{aligned}$$



with  $f_{0,s} = m_s, f_{C+1,s} = m_{s+1}$ .

- (b) If  $\nu_{h,j}^i = 0$  and  $\mu_i^s \neq 0$ , then the  $\mathcal{C}$ -cell given by

$$\{\bar{x} \in M^n : Z_{i,j}(\bar{x}; \bar{a}, m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{a}), \dots, m_s(\bar{x}, \bar{a}))\}$$

has at most  $C_n$  points, where  $C_n$  is a constant that might be different to the constant  $C$  of the case  $n = 1$ . Let  $\bar{p}_1, \dots, \bar{p}_{C_n}$  be an enumeration of such points. We then consider the refinement given by the collections

$$\{(\bar{p}_r, z) : m_s(\bar{p}_r) < z < m_{s+1}(\bar{p}_r)\} : r \leq C_n \cup \{(\bar{p}_r, z) : m_s(\bar{p}_r) : r \leq C_n\}.$$

Let  $\hat{\mathfrak{X}}(z, \bar{x}; \bar{a})$  be the refinement of  $\mathfrak{X}(z, \bar{x}; \bar{a})$  after the processes (a) and (b) (leaving any cell outside this cases unmodified) which again provides a  $\mathcal{C}$ -cell decomposition of  $M^{n+1}$ . For every  $Z \in \hat{\mathfrak{X}}(z, \bar{x}; \bar{a})$  let  $L_Z$  be a 1- $\mathcal{C}$ -cell contained in  $Z$ , with maximal size. We have the following cases:

- If  $Z$  is obtained after process (a) we consider two cases:

- \* The  $\mathcal{C}$ -cell  $Z$  has the form

$$\{(z, \bar{x}) : Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{a}), \dots, m_k(\bar{x}, \bar{a})) \wedge f_{r,s}(\bar{x}) < z < f_{r+1,s}(\bar{x})\}$$

and the intended measure will be  $\alpha_Z = 0$ . Clearly,

$$\|\varphi(M^{n+1}; \bar{a}) \cap Z\| - \alpha_Z |Z| = 0 \leq C |L_Z|^{\dim Z - 1/2}. \quad \checkmark$$

- \* The  $\mathcal{C}$ -cell  $Z$  has the form

$$\{(z, \bar{x}) : Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{a}), \dots, m_s(\bar{x}, \bar{a})) \wedge f_{r,s}(\bar{x}) = z\}$$

and the intended measure is  $\alpha_Z = \nu_{h,j}^i$ . Note that

$$\dim Z_{i,j} = \dim Z_{i,j} \cap \varphi_i = \dim(f_{r,s}(Z_{i,j} \cap \varphi_i)) = \dim Z.$$

Let  $L_{i,j}$  be a 1- $\mathcal{C}$ -cell contained in  $Z_{i,j}$  of maximal size. Then we obtain

$$\begin{aligned} \|\varphi(M^{n+1}; \bar{a}) \cap Z\| - \alpha_Z |Z| &= \|Z_{i,j} \cap \varphi_i - \nu_{h,j}^i |Z_{i,j}|\| \leq C_n \cdot |L_{i,j}|^{\dim Z_{i,j} - 1/2} \\ &= C_n \cdot |\{(\bar{x}; f_{r,s}(\bar{x}) : \bar{x} \in L_{i,j})\}|^{\dim Z_{i,j} - 1/2} \\ &\leq C_n \cdot |L|^{\dim Z - 1/2}. \quad \checkmark \end{aligned}$$

- If  $Z$  is obtained after process (b) we consider two cases again:

- \* The  $\mathcal{C}$ -cell  $Z$  has the form  $Z = \{(\bar{p}_r, z) : m_s(\bar{p}_r) < z < m_{s+1}(\bar{p}_r)\}$  and is itself a 1- $\mathcal{C}$ -cell. Take  $\alpha_Z = \mu_i^s > 0$  to obtain

$$\begin{aligned} \|\varphi(M^{n+1}; \bar{a}) \cap Z\| - \alpha_Z |Z| &= \|\varphi(M, \bar{p}_r; \bar{a}) \cap (m_s(\bar{p}_r), m_{s+1}(\bar{p}_r)) - \mu_i^s |(m_s(\bar{p}_r), m_{s+1}(\bar{p}_r))|\| \\ &\leq C \cdot |(m_s(\bar{p}_r), m_{s+1}(\bar{p}_r))|^{1/2} \\ &= C \cdot |\{(z, \bar{p}_r) : m_s(\bar{p}_r) < z < m_{s+1}(\bar{p}_r)\}|^{1/2} \\ &= C \cdot |L_Z|^{1/2} = C \cdot |L_Z|^{\dim Z - 1/2}. \quad \checkmark \end{aligned}$$

- \* The  $\mathcal{C}$ -cell  $Z$  has the form  $Z = \{(m_s(\bar{p}_r), \bar{p}_r)\}$ . In this case we can take the measure  $\alpha_Z = 0$ . The desired inequality clearly holds if we consider the fact that the size of the maximal 1- $\mathcal{C}$ -cell contained in a point is 1.

- (c) If  $Z$  did not pass through neither process (a) nor (b), then  $\mu_i^s, \nu_{i,h}^j > 0$ . Again we have two cases:

\* The  $\mathcal{C}$ -cell  $Z$  has the form

$$\{(z, \bar{x}) : Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{a}), \dots, m_k(\bar{x}, \bar{a})) \wedge z = m_s(\bar{a})\}.$$

This case is analogue to the second case in (a), replacing the function  $f_{r,s}$  by  $m_s$ . The intended measure would be  $\alpha_Z = \nu_{i,h}^j$ .

\* The  $\mathcal{C}$ -cell  $Z$  has the form

$$Z = \{(z, \bar{x}) : Z_{i,j}(\bar{x}; m^n(\bar{a})) \wedge \varphi_i(\bar{x}; \bar{a}, m_1(\bar{x}, \bar{a}), \dots, m_k(\bar{x}, \bar{a})) \wedge m_s(\bar{x}, \bar{a}) < z < m_{s+1}(\bar{x}, \bar{a})\}.$$

The intended measure in this case is  $\alpha_Z = \mu_i^s$ . Let  $L$  be the interval  $(m_s(\bar{b}, \bar{a}), m_{s+1}(\bar{b}, \bar{a}))$  with  $\bar{b} \in Z_{i,j} \cap \phi_i$  of maximal size. Note that  $\dim Z = \dim Z_{i,j} + 1$  and we obtain

$$\begin{aligned} & |\varphi(M^{n+1}; \bar{a}) \cap Z| \\ &= \sum_{\bar{b} \in Z_{i,j} \cap \phi_i} |\varphi(M, \bar{b}; \bar{a}) \cap (m_s(\bar{b}, \bar{a}), m_{s+1}(\bar{b}, \bar{a}))| \\ &\leq \sum_{\bar{b} \in Z_{i,j} \cap \phi_i} \left( \mu_i^s \cdot |(m_s(\bar{b}, \bar{a}), m_{s+1}(\bar{b}, \bar{a}))| + C |(m_s(\bar{b}, \bar{a}), m_{s+1}(\bar{b}, \bar{a}))|^{1/2} \right) \\ &\leq \mu_i^s \cdot \left( \sum_{\bar{b} \in Z_{i,j} \cap \phi_i} |(m_s(\bar{b}, \bar{a}), m_{s+1}(\bar{b}, \bar{a}))| \right) + C \cdot |L|^{1/2} |Z_{i,j} \cap \phi_i| \\ &\leq \mu_i^s \cdot |Z| + C_1 |L|^{1/2} \left( \nu_{i,h}^j \cdot |Z_{i,j}| + C_n |L_{i,j}|^{\dim Z_{i,j}-1/2} \right) \\ &\leq \mu_i^s \cdot |Z| + C_1 \cdot \nu_{i,h}^j \cdot |L|^{1/2} |L_{i,j}|^{\dim Z_{i,j}} + C_1 C_n \cdot |L|^{1/2} |L_{i,j}|^{\dim Z_{i,j}-1/2} \\ &\quad \text{(because } |Z_{i,j}| \leq |L_{i,j}|^{\dim Z_{i,j}} \text{)} \\ &\leq \mu_i^s \cdot |Z| + 2C_1 \cdot C_n |L_Z|^{\dim Z_{i,j}+1/2} \\ &\leq \mu_i^s \cdot |Z| + C_{n+1} |L_Z|^{(\dim Z_{i,j}+1)-1/2} = \mu_i^s \cdot |Z| + C_{n+1} |L_Z|^{\dim Z-1/2}. \quad \checkmark \end{aligned}$$

This completes the proof of the condition (1) for the inductive case.

In any case, the formulas  $\chi_{\bar{v}}^i(\bar{a}, m^n(\bar{a}))$  prove that the cells in  $\hat{\mathfrak{X}}(\bar{a}, m^n(\bar{a}))$  are definable. So, the definability condition holds.  $\square$

### 3. EXAMPLES OF $\mathcal{O}$ -ASYMPTOTIC CLASSES

It was already mentioned in Section 2 that the class of finite linear orders is an  $\mathcal{O}$ -asymptotic class. We will present in this section another example of an  $\mathcal{O}$ -asymptotic class.

#### 3.1. Cyclic groups with an ordering.

**Definition 3.1.** *Given a natural number  $N$ , we consider the finite linearly ordered structure  $\mathcal{Z}_N = (\mathbb{Z}/(2N+1)\mathbb{Z}, +, <)$  with the usual additive structure and the linear order defined as:*

$$-\overline{N} < -\overline{N-1} < \dots < \overline{0} < \dots < \overline{N-1} < \overline{N}.$$

We will show that the class  $\mathcal{C} = \{\mathcal{Z}_N : N < \omega\}$  is an  $\mathcal{O}$ -asymptotic class. First, we need a result that describes uniformly the definable sets for this class.

**Lemma 3.2.** *Let  $\mathcal{C}_{\text{cyc}} = \{\mathcal{Z}_N : N < \omega\}$ . Then every  $N$  and  $\bar{a} \in \mathcal{Z}_N$  the formula  $\varphi(x; \bar{a})$  is equivalent in  $\mathcal{Z}_N$  (with a uniform number of disjunctions) to a boolean combination of formulas of the form:*

$$x = b, \quad x < b, \quad x > b, \quad \text{and} \quad P_m(x + b)$$

where  $P_m(y) := \exists t ((0 < t < \dots < mt = y) \vee (0 > t > \dots > mt = y))$  for  $m \geq 2$ .

*Proof.* First, define the function

$$\begin{aligned} f : \mathcal{Z}_N &\longrightarrow \mathbb{Z} \\ \bar{x} &\longmapsto y \end{aligned}$$

where  $y$  is the unique integer such that  $-N \leq y \leq N$  and  $y \equiv x \pmod{2N+1}$ , and the following formulas as an intended interpretation of every  $\mathcal{Z}_N$  into  $(\mathbb{Z}, +, <, 0)$ :

$$Z(x; N) := -N \leq x \leq N$$

$$S(x, y, z) := Z(x) \wedge Z(y) \wedge Z(z) \wedge (x + y = z \vee x + y - N = z \vee x + y + N = z)$$

$$O(x, y) := Z(x) \wedge Z(y) \wedge x < y.$$

Given a formula  $\varphi(x; \bar{y})$  and using these replacements and continuing inductively on the connectives and quantifiers (in the natural way) we obtain a formula  $\hat{\varphi}(x; \bar{y}, w)$  such that  $\mathcal{Z}_N \models \varphi(b; \bar{a})$  if and only if  $\mathbb{Z} \models \hat{\varphi}(f(b); f(\bar{a}), N)$ . By Presburger's Theorem, the formula  $\hat{\varphi}(x; y, w)$  is equivalent in  $\mathbb{Z}$  to a boolean combination of formulas of the form

$$nx = t(\bar{y}, w), \quad nx < t(\bar{y}, w), \quad t(\bar{y}, w) < nx, \quad D_m(nx + t(\bar{y}, w))$$

where  $t(\bar{y}, w)$  is a term which has the form  $\sum_{i=1}^l \alpha_i \cdot y_i + \beta w$  with  $\alpha_i, \beta \in \mathbb{Z}$ , and  $D_m(z)$  is the formula given by  $\exists t(\underbrace{t + t + \dots + t}_{m\text{-times}} = z)$ .

Since  $n$  is fixed, if we replace  $\bar{y}, w$  by  $f(\bar{a}), N$ , the first three formulas produce definable sets which are points or intervals in  $\mathbb{Z} \cap [-N, N]$ , and they will produce boolean combination of points and intervals in  $\mathcal{Z}$  where the maximum number of points and intervals depends only on  $n$ .

Now, let  $\phi(x) := D_m(nx + t(f(\bar{a}), N))$  and assume that  $\mathbb{Z} \models \phi(x)$ . We can write  $t(f(\bar{a}), N) = mq + r$  for  $0 \leq r < m$ . So, there is some  $z \in \mathbb{Z}$  such that  $mz = nx + r$ . We have three cases:

- $g.c.d.(n, m) = 1$ : there is a combination such that  $\gamma_1 n + \gamma_2 m = 1$  and by writing  $\gamma_1 r = mk + r_2$  (with  $0 \leq r_2 < m$ ) we obtain:

$$\begin{aligned} \mathbb{Z} \models D_m(nx + t(f(\bar{a}), N)) &\Leftrightarrow mz = nx + r \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow x \equiv -\gamma_1 r \pmod{m} \\ &\Leftrightarrow x + mk + r_2 \equiv 0 \pmod{m} \\ &\Leftrightarrow x + r_2 \equiv 0 \pmod{m} \end{aligned}$$

So, the definable set  $f^{-1}(\phi(\mathbb{Z})) \subseteq \mathcal{Z}_N$  is the set of realizations of the formula  $P_m(x + f^{-1}(r_2))$ , adding at most one point at each end.

- $g.c.d.(n, m) = h > 1$  and  $h|r$ : In this case we can divide both sides by  $h$  obtaining

$$\begin{aligned} \mathbb{Z} \models D_m(nx + t(f(\bar{a}), N)) &\Leftrightarrow mz = nx + r \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow \left(\frac{m}{h}\right)z = \left(\frac{n}{h}\right)x + \left(\frac{r}{h}\right) \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow m_1 z = n_1 x + r_1 \text{ for some } z \in \mathbb{Z} \end{aligned}$$

where  $m_1 = \frac{m}{h}$ ,  $n_1 = \frac{m}{h}$  and  $r_1 = \frac{r}{h}$ . Since  $\text{m.c.d.}(m_1, n_1) = 1$ , we can continue as in the case above, proving that  $f^{-1}(\phi(\mathbb{Z}))$  is the set of realizations of the formula  $P_{m_1}(x + f^{-1}(r_2))$  (where  $r_2$  is chosen as above from  $m_1, n_1, r_1$ ), adding at most one point at each end.

- $\text{g.c.d.}(n, m) = h > 1$  and  $h \nmid r$ : In this case there is no solution to the equation  $nx + r \equiv 0 \pmod{m}$ , and  $f^{-1}(\phi(\mathbb{Z})) = \emptyset$ .

Therefore, the formula  $\varphi(x; \bar{a})$  is equivalent in  $\mathcal{Z}_N$  (for large enough  $N$ ) to a combination of formulas of the form

$$x = b, \quad x < b, \quad x > b, \quad P_m(x + b),$$

with a bound in the number of intervals and points given by the number of disjunctions in the formula equivalent to  $\hat{\varphi}(x, \bar{y}, w)$  provided by Presburger's Theorem.  $\square$

**Proposition 3.3.** *The class  $\mathcal{C}_{\text{cyc}}$  is an  $\mathcal{O}$ -asymptotic class.*

*Proof.* Let  $\varphi(x; \bar{y})$  be a formula in the language  $\mathcal{L} = \{+, <\}$ . By the Lemma 3.2,  $\varphi(x; \bar{y})$  is equivalent (uniformly in  $\mathcal{C}_{\text{cyc}}$ ) to a boolean combination of formulas of the form  $x = z$ ,  $x < z$  or  $P_m(x + z)$  where  $z$  is a term depending on  $\bar{y}$ . This boolean combination can be assumed to be of the form

$$\bigvee_{0 \leq i \leq k} \left( a_i < x < b_i \wedge \bigwedge_j (P_{m_{i,j}}(x + c_{i,j}))^{\eta_{i,j}} \right)$$

where  $\eta_j$  is either 0 or 1 (with the notation  $\phi^0 = \phi$ ,  $\phi^1 = \neg\phi$ ) and  $b_i \leq a_{i+1}$  for  $i \leq k-1$ .

The decomposition is then given by  $c_{2i} = a_i$ ,  $c_{2i+1} = b_i$  and the possible measures on each interval are the elements  $\{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1\}$  where  $m$  is the product of all the integers  $m_{i,j}$ .  $\square$

**3.2. An example of a non- $\mathcal{O}$ -asymptotic class.** Let  $M_n$  the finite structure given by  $M_n = ([1, n \cdot 2^n], <, P)$  where  $<$  is the usual order on  $\mathbb{N}$  and  $P$  is a unary predicate interpreted as follows:

$$P(M_n) = \bigcup_{k=0}^{n-1} \left[ k \cdot 2^n + 1, k \cdot 2^n + \frac{2^n}{2^k} \right]$$

We will show that the class  $\mathcal{C}_P = \{M_n : n < \omega\}$  is not an  $\mathcal{O}$ -asymptotic class.

Suppose  $\mathcal{C}_P$  is  $\mathcal{O}$ -asymptotic, and let  $\bar{c}^n = (c_0^n, \dots, c_k^n)$  be the tuple in  $M_n$  witnessing the condition (1) for the formula  $P(x)$ . Let  $\mu = \min\{\mu_i > 0 : \bar{\mu} \in E\}$  and take  $n$  is large enough so that  $n > k \cdot \left(\frac{1+C}{\mu}\right)^2$ .

Since there are only  $k$  intervals, one of those intervals has size  $|(c_i, c_{i+1})| = L \geq \frac{n}{k} 2^n$ . Then,

$$\frac{1}{\frac{n}{k}} + \frac{C}{\left(\frac{n}{k}\right)^{1/2}} < (1+C) \frac{1}{\left(\frac{n}{k}\right)^{1/2}} < (1+C) \frac{\mu}{1+C} = \mu$$

and we have

$$\begin{aligned} \frac{|P(x) \cap (c_i, c_{i+1})|}{|(c_i, c_{i+1})|} &< \frac{\sum_{k=1}^{L/2^n} 2^n \cdot \frac{1}{2^k}}{L} \leq \frac{2^n \sum_{k=1}^{L/2^n} \frac{1}{2^k}}{2^n \frac{n}{k}} \\ &\leq \frac{1}{\frac{n}{k}} < \mu - \frac{C}{\left(\frac{n}{k}\right)^{1/2}} < \mu - \frac{C}{|(c_i, c_{i+1})|^{1/2}} \end{aligned}$$

So,

$$|P(x) \cap (c_i, c_{i+1})| < \mu|(c_i, c_{i+1})| - C|(c_i, c_{i+1})|^{1/2}$$

a contradiction.  $\square$ .

In Section 6 we will give some results towards new examples of  $\mathcal{O}$ -asymptotic classes.

#### 4. ULTRAPRODUCTS IN $\mathcal{O}$ -ASYMPTOTIC CLASSES

In this section we present a series of results which will allow us to place the infinite ultraproducts of structures in  $\mathcal{O}$ -asymptotic classes in the classification theory map, i.e., we will study model theoretic properties such as  $\mathcal{O}$ -minimality, NIP, NTP<sub>2</sub>, rosiness for the non-trivial ultraproducts of  $\mathcal{O}$ -asymptotic classes.

**Proposition 4.1.** *Let  $\mathcal{C}$  be a class of finite ordered structures and suppose that every infinite ultraproduct is  $\mathcal{O}$ -minimal. Then  $\mathcal{C}$  is an  $\mathcal{O}$ -asymptotic class*

*Proof.* Let  $\mathcal{C} = \{M_i : i \in \omega\}$  and assume  $\prod_{\mathcal{U}} M_i$  is  $\mathcal{O}$ -minimal for all  $\mathcal{U}$ . We start with an easy claim.

*Claim:* For each formula  $\varphi(x; \bar{y})$  there is a bound  $k$  in the alternation number of  $\varphi(x, \bar{a})$ , along all tuples  $\bar{a}$  in structures of the class  $\mathcal{C}$ .

*Proof of the claim:* Assume not. Then for every  $k < \omega$  there are  $M_{i_k}$  and  $\bar{a}_k \in M_{i_k}$  such that

$$M_{i_k} \models \exists x_0, x_1, \dots, x_{2k+1} \left( \bigwedge_{i=0}^k \varphi(x_{2i}, \bar{a}_k) \wedge \bigwedge_{i=0}^k \neg \varphi(x_{2i+1}, \bar{a}_k) \right).$$

If  $\mathcal{U}$  is a non-principal ultrafilter on  $\omega$  containing the set  $A = \{i_k : k < \omega\}$ , then for  $M = \prod_{\mathcal{U}} M_i$  and the tuple  $\bar{a} = [\bar{a}_k]$  we obtain that

$$M \models \exists x_0, x_1, \dots, x_{2k+1} \left( \bigwedge_{i=0}^k \varphi(x_{2i}, \bar{a}_k) \wedge \bigwedge_{i=0}^k \neg \varphi(x_{2i+1}, \bar{a}_k) \right)$$

for every  $k < \omega$ , contradicting the  $\mathcal{O}$ -minimality of  $M$ , since  $\varphi(M, \bar{a})$  is not a finite union of intervals and points.  $\checkmark$

Let  $\varphi(x; \bar{y})$  be a formula and  $k, m$  the corresponding bounds provided by the claim above. Define

$$\Phi(x, \bar{y}, \bar{z}) := \varphi(x; \bar{y}) \leftrightarrow \left( \bigvee_{i=0}^k z_i < x < z_{i+1} \vee \bigvee_{i=k+1}^l x = z_i \right).$$

Then, for every infinite ultraproduct  $M$  of structures in  $\mathcal{C}$ ,

$$M \models \forall \bar{y} \exists z_0, \dots, z_k, z_{k+1}, \dots, z_{k+l} \forall x (\Phi(x; \bar{y}, \bar{z})).$$

In particular, there is  $N_\varphi \in \mathbb{N}$  such that  $M_i$  satisfies the same sentence for every  $i \geq N_\varphi$  (if not, we can construct an infinite ultraproduct in which the sentence does not hold).

So, we can take  $C = N_\varphi$ ,  $E = \{\bar{\mu} \subseteq [0, 1]^k : \mu_i \in \{0, 1\}\}$  and for every  $\bar{a} \in M$ , we can take  $\bar{c} = (c_0, \dots, c_{k+1})$  the corresponding tuple witnessing  $M \models \forall x (\Phi(x; \bar{a}, \bar{c}))$ . This shows that the class  $\mathcal{C}$  satisfies the condition (1) of the definition.

For the condition (2) (the definability clause), it is enough to take the formulas

$$\psi_{\bar{\mu}}(\bar{y}; \bar{z}) := \forall x (\Phi(\bar{y}, \bar{z})).$$

$\square$

The previous result is not true if we replace the condition of  $\mathcal{O}$ -minimality by quasi- $\mathcal{O}$ -minimality. To show this, consider the class  $\mathcal{C}_P$  defined in Section 3.2. Every ultraproduct of elements in  $\mathcal{C}_P$  is quasi-o-minimal as it follows from the quantifier elimination for  $\mathcal{C}_P$  done in Section 7.2 and Theorem 3 in [1].

Since the leading idea in the definition of  $\mathcal{O}$ -asymptotic classes is that they are melding properties from one-dimensional classes (whose ultraproducts are known to be simple and unstable in general) and  $\mathcal{O}$ -minimal theories (which are known to be unstable theories with NIP), we are not expecting the ultraproducts of  $\mathcal{O}$ -asymptotic classes to be neither simple nor with NIP. The two natural contexts which extends both simple and  $\mathcal{O}$ -minimal theories are rosy theories and theories with NTP<sub>2</sub>. We will show that both of these properties are satisfied by the ultraproducts of  $\mathcal{O}$ -asymptotic classes.

**Theorem 4.2.** *Let  $\mathcal{C}$  be an  $\mathcal{O}$ -asymptotic class. Then for every infinite ultraproduct  $M$  of elements of  $\mathcal{C}$ ,  $\text{Th}(M)$  is superrosy of  $U^b$ -rank 1.*

*Proof.* Let  $M = \prod_{i \in \mathcal{U}} M_i$  be an infinite ultraproduct of structures in an  $\mathcal{O}$ -asymptotic class  $\mathcal{C}$ . To

prove that  $M$  is super rosy of  $U^b$ -rank 1, it is enough to show that the only formulas  $\phi(x, b)$  which  $b$ -fork over the empty set are the algebraic formulas.

Assume otherwise. Then there is a tuple of parameters  $e$  and a formula  $\theta(y, e) \in tp(b/c)$  such that the set

$$\{\phi(x, b') : b' \models \theta(y, e)\}$$

is  $k$ -inconsistent for some  $k < \omega$ .

Put  $b = [b_i]_{i \in \mathcal{U}}$ . Since  $\mathcal{C}$  is an  $\mathcal{O}$ -asymptotic, there is a finite set  $E \subseteq [0, 1]^{l+1}$  and a constant  $C$  (both associated with the formula  $\phi(x, y)$ ) such that for every  $i < \omega$  there is a tuple  $\bar{\mu} \in E$  and  $c_1^{b_i} < c_2^{b_i} < \dots < c_l^{b_i}$  satisfying:

- (1) Either  $\mu_j = 0$  and  $|\phi(M_i, b_i) \cap (c_j^{b_i}, c_{j+1}^{b_i})| \leq C$ , or  $\mu_j > 0$  and

$$||\phi(x, b_i)| - \mu_j|(c_j^{b_i}, c_{j+1}^{b_i})| \leq C|(c_j^{b_i}, c_{j+1}^{b_i})|^{1/2}.$$

- (2) There is a formula  $\phi_{\bar{\mu}}(y, z_1, \dots, z_l)$  such that

$$M_i \models \phi_{\bar{\mu}}(b_i, c_1, \dots, c_l) \quad \text{implies} \quad (1) \text{ holds for } \bar{\mu} \text{ and } b_i, c_1, \dots, c_l.$$

Furthermore, there is a tuple  $\bar{\mu} \in E$  which works for  $b_i$  in an  $\mathcal{U}$ -large set.

Take  $\theta(y, e)$  to imply the formula

$$\exists^{>C \cdot l} x(\phi(x, y)) \wedge \exists z_1, \dots, z_l(\phi_{\bar{\mu}}(y, z_1, \dots, z_l)).$$

The first part of the conjunction implies that for  $b' \models \theta(y, e)$ ,  $\phi(x, b')$  is not algebraic. The second part ensures that the measure  $\bar{\mu}$  works for  $\phi(x, b'_i)$  and suitable  $c_1^{b'_i}, \dots, c_l^{b'_i}$  for an  $\mathcal{U}$ -large set  $I_{b'}$ . Since  $\phi(x, b)$  is non-algebraic,  $\bar{\mu} \neq \vec{0}$ . Let  $\mu_j$  be the first non-zero coordinate of the tuple  $\bar{\mu}$ .

**Claim:** *There is an infinite interval  $(\alpha, \beta) \subseteq (c_j^{b'}, c_{j+1}^{b'})$  for infinitely many  $b' \models \theta(y, e)$ .*

*Proof of the Claim:* Consider the formula

$$L(w) := \exists y, z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_l(\theta(y, e) \wedge \phi_{\bar{\mu}}(y; z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_l)).$$

This formula defines all the left end points of the intervals  $(c_j^{b'}, c_{j+1}^{b'})$  for  $b' \models \theta(y, e)$ . Furthermore, we may assume that different elements of  $L(M)$  correspond to different elements in  $\theta(M, e)$ , using the function  $f_\phi^j(y)$  corresponding to the  $j$ -th position of the minimal tuple that

provides the decomposition for  $\phi(x, b')$  through the structures  $M_i$ .

If  $L(M)$  is finite, there is an element  $c \in M$  such that  $M \models \chi(b', c)$  for infinitely many  $b' \models \theta(y, e)$ , where  $\chi(y, w)$  is the formula

$$\chi(y, w) := \exists z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_l (\phi_{\bar{\mu}}(y; z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_l)).$$

We know that the type  $p(w')$  given by

$$\begin{aligned} & \{\exists z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_l (\phi_{\bar{\mu}}(b'; z_1, \dots, z_{j-1}, c, z_{j+1}, \dots, z_l) \wedge w' < z_{j+1}) : b' \models \chi(y, c)\} \\ & \cup \{S^n(c) < w' : n < \omega\} \end{aligned}$$

is finitely satisfiable. By compactness, there is some  $\beta \models p(w')$ , and so  $(\alpha, \beta)$  with  $\alpha = c$  is the desired interval.

Assume now that  $L(M)$  is infinite. We can conclude (using the same argument that we used before in the proof) that for an  $\mathcal{U}$ -large set of indices  $i$ , there is a decomposition  $d_1^i, \dots, d_{k_L}^i$  in  $M_i$  such that

$$||L(M) \cap (d_h^i, d_{h+1}^i)| - \nu_h|(d_h^i, d_{h+1}^i)|| \leq C|(d_h^i, d_{h+1}^i)|^{1/2}.$$

Note that for every element  $c$  in  $L(M) \cap (d_h, d_{h+1})$ , there is an element  $r_c$  such that  $(c, r_c)$  is infinite and

$$M \models \exists z_1, \dots, z_{j-1}, z_{j+2}, \dots, z_l (\phi_{\bar{\mu}}(b'; z_1, \dots, z_{j-1}, c, r_c, z_{j+2}, \dots, z_l)).$$

Take  $\hat{c}_0 = \min(L(M) \cap (d_h, \max M))$  and set  $\hat{d}_0 = \min\{r_{\hat{c}_0}, d_{h+1}\}$ . Note that the interval  $(\hat{c}_0, \hat{d}_0)$  is infinite.

Now assume  $\hat{c}_m, \hat{d}_m$  have been already constructed satisfying  $(\hat{c}_m, \hat{d}_m)$  is infinite.

Since  $(\hat{c}_m, \hat{d}_m)$  is infinite, from the uniform distribution of the set  $L(w)$  in the intervals  $(d_h^i, d_{h+1}^i)$  and the fact that  $\nu_h > 0$ , we can conclude that  $L(M) \cap (\hat{c}_m, \hat{d}_m)$  is infinite. Let  $\hat{c}_{m+1}$  be the minimum element of this set, and  $\hat{d}_{m+1} = \min\{\hat{d}_m, r_{\hat{c}_{m+1}}\}$ .

$(\hat{c}_{m+1}, \hat{d}_{m+1})$  is infinite: By construction,  $(\hat{c}_{m+1}, r_{\hat{c}_{m+1}})$  is infinite. On the other hand, since  $L(M) \cap (\hat{c}_m, \hat{d}_m)$  is infinite then  $(\hat{c}_{m+1}, \hat{d}_m) \supseteq L(M) \cap (\hat{c}_m, \hat{d}_m) - \{\hat{c}_{m+1}\}$  is infinite.

Since  $\hat{c}_m \in L(M)$ , there is  $b'_m \models \theta(y, e)$  such that  $c_j^{b'_m} = \hat{c}_m$ . By construction,  $(\hat{c}_m, \hat{d}_m) \subseteq (\hat{c}_m, r_{\hat{c}_m}) = (c_j^{b'_m}, c_{j+1}^{b'_m})$ .

Consider now the type given by:

$$q(u, v) := \{\hat{c}_m < u < S^m(u) < v < \hat{d}_m : m < \omega\}.$$

Clearly,  $q(u, v)$  is finitely satisfiable and since  $M$  is  $\aleph_1$ -saturated, there is a pair  $(\alpha, \beta) \models q(u, v)$  in  $M$ . Therefore,  $(\alpha, \beta)$  is infinite and  $(\alpha, \beta) \subseteq (\hat{c}_m, \hat{d}_m) \subseteq (c_j^{b'_m}, c_{j+1}^{b'_m})$ . This completes the proof of the claim.  $\checkmark$

Now, consider the counting measure on  $M$  localized in  $(\alpha, \beta)$ , i.e., the measure given by

$$meas(X) = \lim_{i \rightarrow \mathcal{U}} \frac{|X \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|}.$$

Let  $\{b^t = [b_i^t] : t < \omega\}$  be a set of infinitely many in the ultraproduct such that  $(\alpha, \beta) \subseteq (c_j^{b^t}, c_{j+1}^{b^t})$  and  $b^t \models \theta(y, e)$ , provided by the claim above. For every  $t < \omega$ , we have that

$$||\phi(x, b_i^t) \cap (\alpha_i, \beta_i)| - \mu_j|(\alpha_i, \beta_i)|| \leq C|(\alpha_i, \beta_i)|^{1/2} \quad \text{for } i \text{ in an } \mathcal{U}\text{-large set,}$$

in particular,

$$\begin{aligned} |\phi(x, b_t) \cap (\alpha_i, \beta_i)| &\geq \mu_j |(\alpha_i, \beta_i)| - C |(\alpha_i, \beta_i)|^{1/2} \\ \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \mu_j - C |(\alpha_i, \beta_i)|^{-1/2}. \end{aligned}$$

Since  $(\alpha, \beta)$  is infinite, we can take  $i$  large enough so that  $\frac{C}{|(\alpha_i, \beta_i)|^{1/2}} \leq \frac{\mu_j}{2}$ , obtaining

$$\begin{aligned} \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \mu_j - \frac{\mu_j}{2} \\ \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \frac{\mu_j}{2} \\ \text{meas}(\phi(x, b_t)) &= \lim_{i \rightarrow \mathcal{U}} \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} \geq \frac{\mu_j}{2}. \end{aligned}$$

Therefore, the sets  $\langle \phi(x, b_t) : t < \omega \rangle$  are events in a probability space with  $\text{meas}(\phi(x, b_t)) \geq \epsilon = \frac{\mu_j}{2}$ , then by Fact 1.9, there are  $b_{t_1}, \dots, b_{t_k}$  with  $t_1 < \dots < t_k$  such that

$$\text{meas} \left( \bigcap_{i=1}^k \phi(x, b_{t_i}) \right) \geq \epsilon^{3^{k-1}}$$

in particular,  $\{\phi(x, b_t) : t < \omega\}$  is not  $k$ -inconsistent, and neither is the set  $\{\phi(x, b') : b' \models \theta(y, e)\}$ . Contradiction.  $\square$

**Theorem 4.3.** *Every infinite ultraproduct of members of an  $\mathcal{O}$ -asymptotic class has  $NTP_2$ .*

*Proof.* Let  $M$  be an infinite ultraproduct of members in an  $\mathcal{O}$ -asymptotic class, and suppose that the formula  $\phi(x, \bar{y})$  with  $|x| = 1$  witness  $TP_2$  in  $M$ . In particular, by Fact 1.5 there are mutually indiscernible sequences  $\langle \bar{a}_i : i < \omega \rangle, \langle \bar{b}_i : i < \omega \rangle$  such that:

- The sets  $\{\phi(x; \bar{a}_i) : i < \omega\}$  and  $\{\phi(x; \bar{b}_i) : i < \omega\}$  are both 2-inconsistent.
- For every  $i, j < \omega$ ,  $\phi(M; \bar{a}_i) \cap \phi(M; \bar{b}_j) \neq \emptyset$ .

For every formula  $\phi(x; \bar{a}_i)$ , there is a tuple of measures  $\bar{\mu} \in E_\phi$  such that  $\phi(M; \bar{a}_i)$  admits a decomposition with proportions  $\bar{\mu}$ . By the pigeonhole principle, there are tuples  $\bar{\mu}, \bar{\nu} \in E$  such that  $\phi(M; \bar{a}_i)$  admits a decomposition with proportion  $\bar{\mu}$  for infinitely many  $i < \omega$ , and  $\phi(M; \bar{b}_j)$  admits a decomposition with proportion  $\bar{\nu}$  for infinitely many  $j < \omega$ . Without loss of generality, we can restrict the indiscernibles sequences to such indices.

Note that for every  $i < \omega$  the decomposition of  $\phi(M; \bar{a}_i)$  is given by elements  $c_0^i < \dots < c_k^i$ , that can be assumed (as in Section 2) to be the images of the definable functions  $m_t^{\bar{\mu}} := f_t(\bar{a}_i) = c_t^i$  for  $t \leq k$ . Likewise, the decomposition of  $\phi(M; \bar{b}_j)$  is given by the element  $g_t(\bar{b}_j)$  for  $t \leq k$ .

*Claim 1:* Assume  $\mu_t = 0$  for some  $t < k$ . Then for every  $j < \omega$  the intersection  $\phi(M, \bar{b}_j) \cap \phi(M; \bar{a}_i) \cap (f_t(\bar{a}_i), f_{t+1}(\bar{a}_i))$  is empty.

*Proof of the Claim 1:* Otherwise, by mutual indiscernibility we have  $\bar{b}_j \models \exists x(\phi(x; \bar{y}) \wedge \phi(x; \bar{a}_i) \wedge f_t(\bar{a}_i) < x < f_{t+1}(\bar{a}_i))$  for every  $j < \omega$ . However, since  $\mu_t = 0$ ,  $|\phi(M; \bar{a}_i) \cap (f_t(\bar{a}_i), f_{t+1}(\bar{a}_i))| \leq C$  and this contradicts 2-inconsistency of  $\{\phi(x; \bar{b}_j) : j < \omega\}$ . Note that we similarly get the following: if  $\nu_t = 0$  for some  $t < k$ , then for every  $i < \omega$  we have  $\phi(M; \bar{a}_i) \cap \phi(M; \bar{b}_j) \cap (g_t(\bar{b}_j), g_{t+1}(\bar{b}_j)) = \emptyset$ .  $\checkmark$



Consider  $t, l$  minimal such that  $\phi(M; \bar{a}_i) \cap \phi(M; \bar{b}_j) \cap (f_t(\bar{a}_i), f_{t+1}(\bar{a}_i)) \neq \emptyset$  and  $\phi(M; \bar{a}_i) \cap \phi(M; \bar{b}_j) \cap (g_l(\bar{b}_j), g_{l+1}(\bar{b}_j)) \neq \emptyset$ . By the claim above,  $\mu_t = \mu$  and  $\nu_l = \nu$  are both positive.

*Claim 2:* For  $i_1 < i_2 < \omega$ ,  $(f_t(\bar{a}_{i_1}), f_{t+1}(\bar{a}_{i_1})) \cap (f_t(\bar{a}_{i_2}), f_{t+1}(\bar{a}_{i_2})) = \emptyset$ .

*Proof of Claim 2:* Assume otherwise, and put  $R = \frac{1}{\mu}$ . By indiscernibility we have that for every  $i < j < \omega$  the intersection  $(f_t(\bar{a}_i), f_{t+1}(\bar{a}_i)) \cap (f_t(\bar{a}_j), f_{t+1}(\bar{a}_j))$  is non-empty. Thus, the intersection

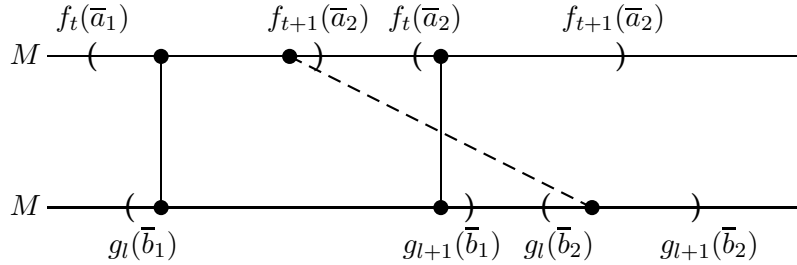
$$\bigcap_{i=1}^{R+1} (f_t(\bar{a}_i), f_{t+1}(\bar{a}_i)) = (f_t(\bar{a}_i), f_{t+1}(\bar{a}_i)) \cap (f_t(\bar{a}_j), f_{t+1}(\bar{a}_j)) := (u, v) \neq \emptyset.$$

By the uniform distribution of  $\phi(M; \bar{a}_i)$  along  $(u, v)$  we have that, if  $meas$  is normalized counting measure with respect to  $(u, v)$ ,  $meas(\phi(M; \bar{a}_i)) = \mu > 0$ , and

$$meas \left( \bigcup_{i=1}^{R+1} \phi(M; \bar{a}_i) \right) = \sum_{i=1}^{R+1} meas(\phi(M; \bar{a}_i)) = (R+1) \cdot \mu > 1,$$

which is a contradiction.  $\checkmark$

Note that a similar conclusion can be obtained for the intervals  $(g_l(\bar{b}_i), g_{l+1}(\bar{b}_i))$ . So, without loss of generality, we may assume that  $f_{t+1}(\bar{a}_1) < f_t(\bar{a}_2)$  and  $g_{l+1}(\bar{b}_1) < g_l(\bar{b}_2)$ , but this contradicts the fact that  $(f_t(\bar{a}_2), f_{t+1}(\bar{a}_2)) \cap (g_l(\bar{b}_1), g_{l+1}(\bar{b}_1)) \neq \emptyset$  as we can see in the following diagram:



□

## 5. RECOVERING $\mathcal{O}$ -MINIMALITY FROM $\mathcal{O}$ -ASYMPTOTIC CLASSES

Let  $\{M_i : i \in I\}$  be a collection of finite linearly ordered structures which contains structures of arbitrarily large cardinality, and let  $M = \prod_{\mathcal{U}} M_i$  be its ultraproduct with respect to a non-principal ultrafilter  $\mathcal{U}$  on  $I$ . Results in Section 3 and 4 suggest that most of these ultraproducts are far from being  $\mathcal{O}$ -minimal.

In this section we consider the problem to obtain dense  $\mathcal{O}$ -minimal structures from the ultraproducts of  $\mathcal{O}$ -asymptotic classes. Every infinite ultraproduct of structures in an  $\mathcal{O}$ -asymptotic class is discrete, with order-type  $\omega \oplus (\kappa \times \mathbb{Z}) \oplus \omega^*$ , where  $\kappa$  is a dense linear order without end points. However, there are two (non-definable) ways to obtain dense linear orders (with end points) as quotients of these ultraproducts. We will present both constructions, each with some preliminary results that suggest there might be some  $\mathcal{O}$ -minimal properties in these quotients.

The first of the two constructions has as image the real unit interval  $[0, 1]$ .

**Construction 5.1.** If  $|M_i| = k_i + 1 \in \mathbb{N}$ , we may assume the universe of  $M_i$  to be  $[0, k_i] = \{0, 1, 2, \dots, k_i\}$ . We can define the map  $\gamma_i$  to be:

$$\begin{aligned} \gamma_i : M_i = [0, k_i] &\longrightarrow [0, 1) \subseteq \mathbb{R} \\ j &\longmapsto \frac{j}{|M_i|} = \frac{j}{k_i + 1}. \end{aligned}$$

These functions induce a map  $\gamma = \prod_{\mathcal{U}} \gamma_i : M \longrightarrow [0, 1)^*$  (where the interval  $[0, 1)^*$  is considered in the non-standard real field  $\mathbb{R}^*$ ). By taking the standard map we define  $\hat{\gamma}(c) = st(\gamma(c))$  for  $c \in M$ , obtaining the following diagram:

$$\begin{array}{ccc} & & [0, 1] \subseteq \mathbb{R} \\ & \nearrow \hat{\gamma} & \uparrow st \\ M = \prod_{\mathcal{U}} M_i & \xrightarrow{\gamma} & [0, 1)^* \subseteq \mathbb{R}^* \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ M_i & \xrightarrow{\gamma_i} & [0, 1) \end{array}$$

**Remark 5.2.** Note we can obtain the map  $\hat{\gamma}$  also with the equation

$$\hat{\gamma} = st \left( \lim_{i \rightarrow \mathcal{U}} \frac{|\{x \in M_i : x \leq a\}|}{|M_i|} \right) = st(meas([\min M, a]))$$

where  $meas$  is the normalized counting measure on  $M$ .

The following is a straightforward result:

**Proposition 5.3.** *The map  $\hat{\gamma}$  is surjective.*

*Proof.* We start this proof by showing the following:

*Claim:* The set  $\hat{\gamma}(M)$  is dense in  $[0, 1)$ .

*Proof of the Claim:* Let  $\alpha \in [0, 1)$  and  $\epsilon > 0$  such that  $(\alpha - \epsilon, \alpha + \epsilon) \subseteq [0, 1)$ . Since  $\mathcal{U}$  is a non-principal ultrafilter on a countable index set (that we may identify with  $\omega$ ), there is a increasing sequence  $\{k_n : n < \omega\}$  of natural numbers and a subsequence of structures  $\{M_{i_n} : n < \omega\}$  such that  $|M_{i_n}| = k_n + 1$  and  $I = \{i_n : n < \omega\} \in \mathcal{U}$ . Take  $m_0 = \min \left\{ k_n : \frac{1}{k_n + 1} < \frac{\epsilon}{2} \right\}$ . It is clear that  $\{i_n : k_n \geq m_0\} \in \mathcal{U}$ .

For every  $n < \omega$ , take  $m_n = \min \left\{ m \in \mathbb{N} : \frac{m}{k_n + 1} > \alpha - \frac{\epsilon}{2} \right\}$ . By the choice of  $k_n, m_n$  and  $\epsilon$  we have  $m_n \leq k_n$ , so

$$\alpha - \frac{\epsilon}{2} < \gamma_{i_n}(S^{m_n}(0)) := \frac{m_n}{k_n + 1} < \frac{m_n + 1}{k_n + 1} < \alpha + \epsilon.$$

Consider the element  $u = (u_i)$  in  $M$  defined by:

$$u_i = \begin{cases} S^{m_n}(0), & \text{if } i = i_n \in I \\ 0, & \text{otherwise.} \end{cases}$$

By Łoś' Theorem we have  $\alpha - \frac{\epsilon}{2} < \gamma(u) < \alpha + \frac{\epsilon}{2}$ , and  $\hat{\gamma} \in (\alpha - \epsilon, \alpha + \epsilon)$ .  $\checkmark$

Now we prove that  $\hat{\gamma}$  is surjective. Take  $\alpha \in [0, 1]$ . Since  $\hat{\gamma}(M)$  is dense in  $[0, 1]$ , there is an increasing sequence  $\{\hat{\gamma}(u_n) : n < \omega\}$  and a decreasing sequence  $\{\hat{\gamma}(v_n) : n < \omega\}$  such that

$$\lim_{n \rightarrow \infty} \hat{\gamma}(u_n) = \alpha = \lim_{n \rightarrow \infty} \hat{\gamma}(v_n).$$

Consider the type  $p(x) = \{u_n < x : n < \omega\} \cup \{x < v_n : n < \omega\}$ , which has to be realized in  $M$  since  $M$  is  $\aleph_1$ -saturated. Let  $c$  be a realization of  $p(x)$ . Since  $\hat{\gamma}$  is a order-preserving map, we obtain

$$\alpha = \lim_{n \rightarrow \infty} \hat{\gamma}(u_n) \leq \hat{\gamma}(c) \leq \lim_{n \rightarrow \infty} \hat{\gamma}(v_n) = \alpha.$$

□

**Proposition 5.4.** *Let  $\mathcal{C}$  be an  $\mathcal{O}$ -asymptotic class, and  $M$  an infinite ultraproduct of elements of  $\mathcal{C}$ . For every definable  $X \subseteq M^1$ ,  $\hat{\gamma}(X)$  is a finite union of intervals and points.*

*Proof.* Assume  $X = \varphi(M; \bar{a})$  for some  $\mathcal{L}$ -formula  $\varphi(x, \bar{y})$  and some tuple  $\bar{a} = (\bar{a}_i) \in M$ . Let  $C_\varphi > 0$  and  $E_\varphi \subseteq [0, 1]^k$  be the constant and the set of tuples of real numbers associated to the formula  $\varphi$ . There is a fix tuple  $\bar{\mu} \in E_\varphi$  such that, for an  $\mathcal{U}$ -large set of indices  $i$  the following holds: There are elements  $c_1^i < \dots < c_k^i \in M_i$  such that for every  $j \leq k$  either  $\mu_j = 0$  and  $|\varphi(M_i, \bar{a}_i) \cap (c_j^i, c_{j+1}^i)| \leq C$ , or,  $\mu_j \neq 0$  and  $|\varphi(M_i, \bar{a}_i) \cap (c_j^i, c_{j+1}^i)| - \mu_j |(c_j^i, c_{j+1}^i)| \leq C |(c_j^i, c_{j+1}^i)|^{1/2}$ .

Define  $c_j = [c_j^i] \in M$  for  $j \leq k$ . It is enough to show that for every  $j = 1, \dots, k$  the set  $\hat{\gamma}(\varphi(M; \bar{a}) \cap (c_j, c_{j+1}))$  is a finite union of intervals and points. We have three cases:

- $\hat{\gamma}(c_j) = \hat{\gamma}(c_{j+1}) = \alpha$ : In this case,  $\hat{\gamma}(\varphi(M; \bar{a}) \cap (c_j, c_{j+1})) = \{\alpha\}$ .
- $\hat{\gamma}(c_j) < \hat{\gamma}(c_{j+1})$  and  $\mu_j = 0$ : In this case,  $\varphi(M; \bar{a}) \cap (c_j, c_{j+1})$  consists of at most  $C$  points. So does  $\hat{\gamma}(\varphi(M; \bar{a}) \cap (c_j, c_{j+1}))$ .
- $\hat{\gamma}(c_j) < \hat{\gamma}(c_{j+1})$  and  $\mu_j > 0$ : In this case, we will show that

$$\hat{\gamma}(\varphi(M; \bar{a}) \cap (c_j, c_{j+1})) = (\hat{\gamma}(c_j), \hat{\gamma}(c_{j+1})).$$

Let  $\alpha \in (\hat{\gamma}(c_j), \hat{\gamma}(c_{j+1}))$ , and consider sequences  $\langle q_n, r_n : n < \omega \rangle$  in  $(\hat{\gamma}(c_j), \hat{\gamma}(c_{j+1}))$  converging to  $\alpha$ , such that

$$\hat{\gamma}(c_j) < q_1 < \dots < q_n < \dots < \alpha < \dots < r_n < \dots < r_1 < \hat{\gamma}(c_{j+1}).$$

By Proposition 5.3, there are elements  $\langle (u_n, v_n) : n < \omega \rangle$  in  $M$  such that  $\hat{\gamma}(u_n) = q_n$ , and  $\hat{\gamma}(v_n) = r_n$  for  $n < \omega$ . Since  $\hat{\gamma}$  cannot distinguish between two elements in  $M$  whose distance is finite, the intervals  $(u_n, v_n)$  are infinite in  $M$ . Consider the partial type  $p(x) := \{u_n < x : n < \omega\} \cup \{x < v_n : n < \omega\} \cup \{\varphi(x, \bar{a})\}$ . By the uniform distribution of  $\varphi(x, \bar{a})$  (with measure  $\mu_j$ ) we can conclude that  $X \cap (u_n, v_n)$  is non-empty, thus  $p(x)$  is finitely satisfiable. Since  $M$  is  $\aleph_1$ -saturated, there is  $w \models p(x)$  in  $M$ , and we have that

$$\alpha = \lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} \hat{\gamma}(u_n) \leq \hat{\gamma}(w) \leq \lim_{n \rightarrow \infty} \hat{\gamma}(v_n) = \lim_{n \rightarrow \infty} r_n = \alpha.$$

This completes the proof.

□

Note that the images of definable subsets of  $M^1$  under the map  $\hat{\gamma}$  are finite union of points and *closed* intervals.

It is possible to extend the map  $\hat{\gamma}$  to higher dimensional definable sets in a natural way.

**Definition 5.5.**

- (i) For a definable set  $X \subseteq M^n$ , we define the set  $\hat{\gamma}(X)$  to be

$$\hat{\gamma}(X) := \{(\hat{\gamma}(a_1), \dots, \hat{\gamma}(a_n) : (a_1, \dots, a_n) \in X\}.$$

(ii) In particular, for a definable function  $f : M \longrightarrow M$  we define  $\widehat{\gamma}(f)$  to be the set

$$\widehat{\gamma}(f) := \{(\widehat{\gamma}(a), \widehat{\gamma}(f(a))) : a \in M\}.$$

The images of definable subsets of  $M^n$  induce a structure on the interval  $[0, 1]$  by taking the boolean algebra generated by the images of definable subsets of  $M^n$ . A question that arises is the following:

**Question 5.6.** *Under which conditions the function  $\widehat{\gamma} : M \rightarrow [0, 1]$  induces an  $\mathcal{O}$ -minimal structure on the interval  $[0, 1]$ ?*

**Conjecture 5.7.** *Assume  $f : M \longrightarrow M$  is a definable function and  $M = \prod_{\mathcal{U}} M_i$  is a ultraproduct of elements in an  $\mathcal{O}$ -asymptotic class. Then, there are finitely many continuous functions  $f_1, \dots, f_k$  with  $f_i : [0, 1] \longrightarrow [0, 1]$  such that  $\gamma(f) \subseteq \bigcup_{i=1}^k \text{graph}(f_i)$ .*

**Proposition 5.8.** *Let  $f : M \longrightarrow M$  be a definable function. Then for every  $\alpha \in [0, 1]$ , the set  $X_\alpha := \{\beta \in [0, 1] : (\alpha, \beta) \in \widehat{\gamma}(f)\}$  is finite.*

*Proof.* Assume  $X_\alpha$  is infinite for some  $\alpha \in [0, 1]$ . Then there is a sequence of different elements  $\{\beta_i : i < \omega\} \subseteq X_\alpha$ . Without loss of generality, we may assume that  $\langle \beta_i : i < \omega \rangle$  is an increasing sequence.

Take  $a$  and  $\langle a_i, b_i : i < \omega \rangle$  be elements of  $M$  such that:

- $f(a_i) = b_i$
- $\gamma(b_i) = \beta_i$
- $a \neq a_i$  and  $\gamma(a) = \gamma(a_i) = \alpha$ .

Since  $\langle \beta_i : i < \omega \rangle$  is an increasing sequence in the real interval  $[0, 1]$ , we can take  $\beta = \gamma(b)$  to be the limit of the sequence  $\langle \beta_i : i < \omega \rangle$ .

Consider the formula

$$\varphi(y; u, v, w, z) := u < y < v \wedge \exists x (w < x < z \wedge f(x) = y).$$

By  $\mathcal{O}$ -asymptoticity, we can take  $\nu > 0$  minimal between all possible positive numbers appearing in the tuples of  $E_\varphi$ .

Let  $\delta_1, \delta_2$  be elements in  $[0, 1]$  such that  $\delta_1 < \alpha < \delta_2$  and  $\frac{\nu}{2}|(\beta_1, \beta)| > |(\delta_1, \delta_2)|$ , and take  $d_1, d_2$  in  $M$  such that  $\gamma(d_1) = \delta_1$  and  $\gamma(d_2) = \delta_2$ . Consider the formula  $\varphi(y; b_1, b, d_1, d_2)$ . Since we have that  $\varphi(M; b_1, b, d_1, d_2)$  is infinite (it contains all the elements  $b_i$  for  $i \geq 2$ ), it holds for  $\mathcal{U}$ -almost all  $i$  that

$$\begin{aligned} |\varphi(M; b_1^i, b^i, d_1^i, d_2^i)| &\geq \nu \cdot |(b_1^i, b^i)| - C|(b_1^i, b^i)|^{1/2} \\ &\geq \nu \cdot |(b_1^i, b^i)| - \frac{\nu}{2}|(b_1^i, b^i)| \\ &\quad (\text{because } |(b_1^i, b^i)| \geq (2C/\nu)^2 \text{ for } \mathcal{U}\text{-almost all } i) \\ &\geq \frac{\nu}{2}|(b_1^i, b^i)|. \end{aligned}$$

Also, since  $f$  is a function,  $|\varphi(M; b_1^i, b^i, d_1^i, d_2^i)| \leq |(d_1^i, d_2^i)|$  for  $\mathcal{U}$ -almost all  $i$ .

Thus,

$$\frac{|(d_1^i, d_2^i)|}{|M_i|} \geq \frac{|\varphi(M_i; b_1^i, b^i, d_1^i, d_2^i)|}{|M_i|} \geq \frac{\nu}{2} \frac{|(b_1^i, b^i)|}{|M_i|},$$

and taking the limit across the ultraproduct we obtain  $(\delta_2 - \delta_1) \geq \frac{\nu}{2}(\beta_2 - \beta_1)$ , contradicting the choice of  $\delta_1, \delta_2$ .  $\square$

Now we present the second construction, which has as image an  $\aleph_1$ -saturated dense linear order with endpoints.

**Construction 5.9.** Consider the ( $\bigvee$ -definable) equivalence relation  $E$  given by  $xEy$  if and only if there is some  $n < \omega$  such that  $x \in [S^{-n}(y), S^n(y)]$ , that is, if and only if the distance between  $x$  and  $y$  is finite. We consider the projection map  $\pi_E : M \rightarrow M/E$ . Note that  $M/E$  is a dense linear order with endpoints, that we can describe as  $\{-\infty\} \cup \kappa \cup \{+\infty\}$ .

**Proposition 5.10.** *Let  $X$  be a definable subset of  $M^1$ . Then  $\pi_E(X)$  is a finite union of intervals and points.*

*Proof.*  $X$  admits a decomposition into  $l$  intervals, witnessed by  $c_0 < \dots < c_l$ . Let  $\bar{\mu} \in [0, 1]^l$  be the tuple of measures which is recurrent for  $X$  through the ultraproduct. If  $\mu_i = 0$ , then  $X \cap (c_{i-1}, c_i)$  is finite (with at most  $C$  points) and  $\pi_E(X \cap [c_{i-1}, c_i])$  has at most  $C$  points. If  $\mu_i > 0$ , then by uniformity there is  $n$  large enough such that if  $(a, b) \subseteq (c_{i-1}, c_i)$  and  $|(a, b)| \geq n$  then  $X \cap (a, b) \neq \emptyset$ . So, if  $\pi_E(c_{i-1}) < \alpha = \pi_E(a) < \pi_E(c_i)$ , take  $b \in X \cap (a, S^n(a))$  to obtain  $\pi_E(a) = \alpha = \pi_E(b) \in \pi_E(X)$ . Therefore,  $\pi_E(X \cap [c_{i-1}, c_i]) = [\pi_E(c_{i-1}), \pi_E(c_i)]$  and by taking the union we conclude that

$$\pi_E(X) = \bigcup_{i=1}^k \pi_E(X \cap [c_{i-1}, c_i])$$

is a finite union of intervals and points.  $\square$

Again, the images of definable subsets of  $M^1$  under the map  $\pi_E$  are finite union of points and closed intervals.

As before, we can generalize the map  $\pi_E$  to higher dimensional definable sets in a natural way.

**Definition 5.11.**

- (i) For a definable set  $X \subseteq M^n$ , we define the set  $\pi_E(X)$  to be

$$\pi_E(X) := \{(\pi_E(a_1), \dots, \pi_E(a_n)) : (a_1, \dots, a_n) \in X\}.$$

- (ii) In particular, for a definable function  $f : M \rightarrow M$  we define  $\pi_E(f)$  to be the set

$$\pi_E(f) := \{(\pi_E(a), \pi_E(f(a))) : a \in M\}$$

It is possible to induce a structure on  $I = M/E$  by taking the boolean algebra generated by the images of definable subsets of  $M^n$  under  $\pi_E$ . So we can ask the following:

**Question 5.12.** *Under which conditions the structures induced by  $M$  on  $I$  is  $\mathcal{O}$ -minimal?*

**Definition 5.13.** A definable function  $f : M \rightarrow M$  is said to be pseudocontinuous if there is a natural number  $N = N_f$  such that for every  $a \in M$ ,

$$f(S(a)) \in [S^{-N}(f(a)), S^N(f(a))]$$

**Proposition 5.14.** *If  $f$  is a pseudocontinuous function, then  $\pi_E(f)$  is a function on  $M/E$ .*

*Proof.* Assume  $(\alpha, \beta_1), (\alpha, \beta_2) \in \pi_E(f)$ , and take  $a < a' \in M$  such that  $\pi_E(a) = \alpha = \pi_E(a')$ ,  $\beta_1 = \pi_E(f(a))$  and  $\beta_2 = \pi_E(f(a'))$ . Let  $\beta$  be an element in  $M/E$  with  $\beta_1 < \beta < \beta_2$ . Since  $\pi_E(a) = \pi_E(a')$ , we have that there is  $m < \omega$  such that  $a' = S^m(a)$ . If  $N = N_f$  is the natural number provided by the pseudocontinuity of  $f$ , we have that  $f(a') \in [S^{-(N \cdot m)}(f(a)), S^{(N \cdot m)}(f(a))]$ , concluding that  $f(a)Ef(a')$ , that is,  $\beta_1 = \beta_2$ .  $\square$

**Proposition 5.15.** *Let  $f : M \rightarrow M$  be a pseudocontinuous function. Then  $\pi_E(f) : M/E \rightarrow M/E$  is continuous (in the sense of the order topology).*

*Proof.* Put  $g := \pi_E(f)$  and let  $\alpha$  be an arbitrary element of  $M/E$ . Consider an interval  $(\beta_1, \beta_2)$  containing  $g(\alpha)$ . Let  $a, b_1, b_2$  be elements in  $M$  such that  $\pi_E(a) = \alpha, \pi_E(b_1) = \beta_1$  and  $\pi_E(b_2) = \beta_2$  and consider the formulas

$$\begin{aligned}\varphi_1^-(x) &:= x < a \wedge f(x) \geq b_2, & \varphi_2^-(x) &:= x < a \wedge f(x) \leq b_1, \\ \varphi_1^+(x) &:= x > a \wedge f(x) \geq b_2, & \varphi_2^+(x) &:= x > a \wedge f(x) \leq b_1.\end{aligned}$$

Let  $a_1 = \max \varphi_1^-(M) \cup \varphi_2^-(M)$  and  $a_2 = \min \varphi_1^+(M) \cup \varphi_2^+(M)$ . By maximality of  $a_1$  and minimality of  $a_2$  we have that  $a_1 < x < a_2$  implies  $f(x) \in (b_1, b_2)$ . If we prove that  $\pi_E(a_1) < \alpha < \pi_E(a_2)$  we are done. Assume for example that  $\pi_E(a_1) = \alpha$ , then there is a natural number  $k$  such that  $a = S^k(a_1)$ . Since  $f$  is pseudocontinuous, there is a natural number  $N$  such that

$$f(a) \in [S^{-(N \cdot k)}(f(a_1)), S^{(N \cdot k)}(f(a_1))].$$

If  $\beta'_1, \beta'_2$  are elements in  $M/E$  such that  $\beta_1 < \beta'_1 < g(\alpha) < \beta'_2 < \beta_2$ , we would have  $g(\alpha) = \pi_E(f(a)) = \pi_E(f(a_1))$ , and thus  $f(a_1) \in [b_1, b_2]$ , a contradiction.  $\square$

**Conjecture 5.16.** *Assume  $f : M \rightarrow M$  is a definable function and  $M = \prod_{\mathcal{U}} M_i$  is a ultraproduct of elements in an  $\mathcal{O}$ -asymptotic class. Then, there are pseudocontinuous functions  $f_1, \dots, f_k$  with  $f_i : M \rightarrow M$  such that  $\text{graph}(f) \subseteq \bigcup_{i=1}^k \text{graph}(f_i)$ .*

## 6. MORE EXAMPLES

The purpose of this section is to provide some preliminary results towards finding more examples of  $\mathcal{O}$ -asymptotic classes of finite structures.

**Problem 6.1.** *Find an example of an  $\mathcal{O}$ -asymptotic class whose infinite ultraproducts are not NIP.*

In what follows, we will construct the class  $\mathcal{C}_\alpha$  and provide some results towards a proof that the class  $\mathcal{C}_\alpha$  is an  $\mathcal{O}$ -asymptotic class.

Let  $\alpha$  be an irrational real number, with  $0 < \alpha < 1$ , and  $f : \mathbb{N} \rightarrow \mathbb{N}$  be the function given by  $f(x) = \llbracket \alpha \cdot x \rrbracket$ , where  $\llbracket * \rrbracket$  denotes the greatest integer less than or equal to  $*$ .

Consider the class  $\mathcal{C}_\alpha$  given by structures  $M_n = \langle [0, n], <_n, f_n \rangle$  where  $<_n$  is the usual order on  $[0, n]$  and  $f_n(x)$  is the restriction of  $f$  to  $[0, n]$ .

**Lemma 6.2.** *Let  $M$  be a structure in the class  $\mathcal{C}_\alpha$ .*

- (1) *For every  $s < \omega$ , the image of the function  $f^s$  is  $[0, f^s(\max M)]$ .*
- (2) *Take  $N = \llbracket \frac{1}{\alpha} \rrbracket$ . Then for every  $a \in M$ , either  $|f^{-1}(a)| = 0$ ,  $|f^{-1}(a)| = N$  or  $|f^{-1}(a)| = N + 1$ .*

*Proof.* (1) By induction on  $s$ :

- Case  $s = 1$ : Let  $k = \min(M - f(M))$ . We will show that  $k = f(\max M) + 1$ . Since  $f(0) = 0$  we know that  $k > 0$ . By minimality, we have that  $k - 1 = \llbracket \alpha m \rrbracket$  for some  $m \in M$ . Assume such  $m$  to be maximal, then

$$\begin{aligned}k - 1 &= \llbracket \alpha m \rrbracket < k < k + 1 \leq \llbracket \alpha(m + 1) \rrbracket \\ \alpha m &< k < k + 1 \leq \alpha m + \alpha \\ 0 &< k - \alpha m < (k - \alpha m) + 1 \leq \alpha,\end{aligned}$$

a contradiction because  $\alpha < 1$ . Thus, the image of  $f$  is an interval, whose maximum clearly is  $f(\max M)$ .

- Case  $s + 1$ : Assume that the function  $f^s$  has image  $[0, f^s(\max M)]$ , and let  $k$  be minimal in the set  $M - f^{s+1}(M)$ . If  $k < f^{s+1}(\max M)$ , then we have for some maximal  $m < \max M$  that

$$\begin{aligned} k - 1 &= f^{s+1}(m) < k < k + 1 \leq f^{s+1}(m + 1) \\ k - 1 &= \llbracket \alpha \cdot f^s(m) \rrbracket < k < k + 1 \leq \llbracket \alpha \cdot f^s(m + 1) \rrbracket. \end{aligned}$$

By the case  $s = 1$ , there is some  $p$  with  $f^s(m) \leq p \leq f^s(m + 1)$  such that  $k = \llbracket \alpha \cdot p \rrbracket$ . By the induction hypothesis,  $p = f^s(m)$  or  $f^s(m + 1)$  (also, because  $f^s$  is a non-decreasing function). This is a contradiction.

- (2) By (1), if  $a$  is not in  $[0, f(\max M)]$ , then  $f^{-1}(a) = \emptyset$ . Assume now that  $a \leq f(\max M)$  and suppose that  $|f^{-1}(a)| \geq N + 2$ . We have that for some  $k$ ,

$$\begin{aligned} a &= \llbracket \alpha \cdot k \rrbracket = \llbracket \alpha(k + N + 1) \rrbracket < a + 1 \\ a &\leq \alpha k < \alpha k + \alpha(N + 1) < a + 1 \end{aligned}$$

a contradiction, because  $\alpha(N + 1) > 1$ .

Suppose now that  $|f^{-1}(a)| \leq N - 1$ . Then for some  $k$  we have

$$\begin{aligned} \alpha k &< a \leq \alpha(k + 1) < \alpha(k + N - 1) < a + 1 < \alpha(k + N) \\ 0 &< a - \alpha k < \alpha < \alpha(N - 1) < (a - \alpha k) + 1 < \alpha N \end{aligned}$$

a contradiction, because  $\alpha N < 1$ .

So, if  $a \in f(M_n)$ , either  $|f^{-1}(a)| = N$  or  $|f^{-1}(a)| = N + 1$ .

□

**Proposition 6.3.** *The class  $\mathcal{C}_\alpha = M_n : n < \omega$  admits uniform quantifier elimination in the language  $\mathcal{L}' = \{<, S, S^{-1}, \min M, \max M\}$ .*

*Proof.* Note that the atomic formulas in the original language have the form  $f^m x = f^n y$  or  $f^m x < f^n y$  for some  $m, n < \omega$ .

Consider a primitive existential formula of the form

$$\phi := \exists x \left( \bigwedge_i f^{m_i} x = f^{n_i} a_i \wedge \bigwedge_j f^{h_j} x \neq f^{l_j} b_j \wedge \bigwedge_k f^{p_k} x < f^{q_k} c_k \wedge \bigwedge_t f^{r_t} x > f^{s_t} d_t \right)$$

We now give some simplifications of the original formula. First, we can replace the terms which do not contain  $x$  by new parameters (e.g replace  $f^{n_i} a_i$  simply by  $a_j$ ). We also can assume that there are no conjuncts of the form  $f^h x \neq b$  because we can replace them by  $(f^h x < b \vee f^h x > b)$  and since we are dealing with existential formulas, after distributing, we obtain a conjunction of formulas only of the forms  $f^m x = a$ ,  $f^p x < c$  and  $f^r x > d$ .

Since the order is discrete, we have that  $f^p x < c$  if and only if  $\min M \leq f^p x \leq S^{-1}(c)$  and  $f^r x > d$  if and only if  $S(d) \leq f^r x \leq \max M$ .

With these changes, the existential formula has the form

$$\phi = \exists x \left( \bigwedge_{i \leq k} f_i^m x = a_i \wedge \bigwedge_{j \leq \ell} c_j \leq f^{n_j} x \leq d_j \right).$$

Assume, without loss of generality that  $m_1 = \min\{m_1, \dots, m_k\}$  and  $n_1 = \min\{n_1, \dots, n_\ell\}$ .

By Lemma 6.2 we know that for every  $s$ , the range of the function  $f^s$  is  $[0, f^s(\max M)]$ . If  $m_1 \leq n_1$ , then for every  $i \leq k$ ,  $j \leq \ell$  we have that  $f^{m_i} x = a_i$  if and only if  $f^{m_i - m_1}(a_1) = a_i$  and

$c_j \leq f^{n_j}x \leq d_j$  if and only if  $c_j \leq f^{n_j-m_1}(a_1) \leq d_j$ . In this case the formula  $\phi$  is equivalent to

$$\left( 0 \leq a_1 \leq f^m(\max M) \wedge \bigwedge_{i \leq k} f^{m_i-m}(a_1) = a_i \wedge \bigwedge_{j \leq \ell} c_j \leq f^{n_j-m}(a_1) \leq d_j \right).$$

Assume now that  $n_1 < m_1$ , the formula  $\phi$  is equivalent to

$$\left( \bigwedge_{j \leq \ell} (f^{n_j-n_1}(c_1) \leq d_j) \wedge \bigwedge_{i \leq k} (f^{m-m_i}(a_1) = a_i) \wedge \bigwedge_{\{j: n_j \geq m\}} (c_j \leq f^{n_j-m_1}(a_1) \leq d_j) \right).$$

This completes the proof of uniform quantifier elimination in the class  $\mathcal{C}_\alpha$ .  $\square$

Now we start calculating the measure of some formulas that can be defined. First, let start with the formula  $\varphi(x) := f(x) < f(Sx)$ .

*Claim 1:* We have  $\lim_{n \rightarrow \infty} \frac{|\varphi(M_n)|}{|M_n|} = \alpha$ .

*Proof of Claim 1:* Note that every jump in the function corresponds to a realization of  $\varphi(x)$ , and can be uniquely associated to a value in the image of the function  $f$ . This correspondence is almost surjective, except for the value 0 in the image, so

$$\lim_{n \rightarrow \infty} \frac{|\varphi(M_n)|}{|M_n|} = \lim_{n \rightarrow \infty} \frac{\llbracket \alpha \cdot n \rrbracket}{n+1} = \alpha.$$

Moreover,

$$||\varphi(M_n)| - \alpha|M_n|| \leq |(\llbracket \alpha \cdot n \rrbracket) - \alpha \cdot (n+1)| \leq |(\llbracket \alpha \cdot n \rrbracket) - \alpha \cdot n| + |\alpha| \leq 2. \quad \checkmark$$

The uniformity in the distribution follows by taking  $C = N\llbracket \frac{1}{\alpha} \rrbracket$ .  $\checkmark$

Consider now the formula  $\phi(x) := f(x) = f(S^M x)$  for  $M \leq \llbracket \frac{1}{\alpha} \rrbracket$ . To find the limit asymptotic counting measure of  $\phi(M_n)$  as  $n$  tends to infinity, we will use the following more or less known result in number theory, that can be found for example in [16, Chapter 3]:

**Fact 6.4** (Weyl's Equidistribution Theorem). *Let  $\alpha$  be an irrational real number, and consider the sequence  $z_n = \alpha \cdot n - \llbracket \alpha \cdot n \rrbracket$  (the fractional part of  $\alpha \cdot n$ ). Then, for every  $0 \leq a < b \leq 1$ ,*

$$\lim_{n \rightarrow \infty} \frac{|\{k \leq n : a \leq z_k \leq b\}|}{n} = b - a.$$

*Claim 2:* For every  $M$ ,  $f(x) = f(x+M)$  if and only if the fractional part of  $\alpha \cdot x$  is less than  $1 - \alpha M$ .

*Proof of Claim 2:* Assume that  $\alpha x - \llbracket \alpha x \rrbracket < 1 - \alpha M$ . Then we have

$$\llbracket \alpha x \rrbracket < \llbracket \alpha x \rrbracket + \alpha M < \alpha(x+M) < \llbracket \alpha x \rrbracket + 1.$$

This implies that  $f(x) = \llbracket \alpha x \rrbracket = \llbracket \alpha(x+M) \rrbracket = f(x+M)$ . For the other direction, assume that  $\alpha x - \llbracket \alpha x \rrbracket \geq 1 - \alpha M$ . Then  $\alpha x + \alpha M \geq \llbracket \alpha x \rrbracket + 1$  and we obtain  $f(x+M) = \llbracket \alpha(x+M) \rrbracket \geq f(x) + 1 > f(x)$ .  $\checkmark$

Using Claim 2 and Weyl's Equidistribution Theorem, we have that

$$\lim_{n \rightarrow \infty} \frac{\phi(M_n)}{|M_n|} = \lim_{n \rightarrow \infty} \frac{|\{k \leq n : 0 < z_k < 1 - \alpha M\}|}{n} = 1 - \alpha M.$$



## 7. APPENDIX

**Definition 7.1.** Let  $\mathcal{C}$  is a class of finite structures in a language  $\mathcal{L}$ . We say that  $\mathcal{C}$  admits quantifier elimination if for every formula  $\varphi(\bar{x})$  there is a quantifier free formula  $\psi(\bar{x})$  such that for every  $M \in \mathcal{C}$ ,

$$M \models \forall x(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

The fact that the class has uniform quantifier elimination is very different from the fact that every structure in  $M$  has quantifier elimination. The second statement is obvious if we have in mind that they are finite structures and its theory can be captured in a single sentence.

The purpose of this appendix is to show the existence of uniform quantifier elimination (in suitable languages) of some classes that have been used as examples through the current paper.

7.1. Uniform quantifier elimination for  $\mathcal{C}_{ord}$ .

**Definition 7.2.**  $\mathcal{C}_{ord}$  will denote the class of all finite linearly ordered sets, in the language  $\mathcal{L} = \{<\}$ .

Given the language  $\mathcal{L} = \{<\}$  of the class  $\mathcal{C}_{ord}$ , we can expand it to  $\mathcal{L}^* = \{c, C, <, S\}$  where  $c, C$  are constant symbols (which will represent the minimum and the maximum of the structure) and  $S$  is a unary function (which will represent the successor function).

We claim the class  $\mathcal{C}_{ord}$  has uniform quantifier elimination in the language  $\mathcal{L}^*$ :

In this language, the atomic formulas have the form  $S^n(x) = S^m(x)$  or  $S^n(x) < S^m(x)$ . Consider now a primitive formula, which will have the form

$$\exists x \left( \bigwedge_{i < l} t_i < S^{p_i} x \wedge \bigwedge_{j < m} S^{q_j} x < u_j \wedge \bigwedge_{k < n} S^{r_k} x = v_k \right)$$

where  $t_i, q_j, v_k$  are all terms which does not include  $x$  as a variable. Note that we can uniform all the indices  $p_i, q_j, r_k$  just by taking  $N = \max \{p_i, q_j, r_k\}$ , because we know that  $t_i < S^{p_i} x$  is equivalent to  $S^{N-p_i} t_i < S^N x$ . So, we can assume the formula has the form:

$$\exists x \left( \bigwedge_{i < l} t_i < S^N x \wedge \bigwedge_{j < m} S^N x < u_j \wedge \bigwedge_{k < n} S^N x = v_k \right)$$

If  $n \neq 0$ , then the formula is equivalent for either  $x = x$  (if  $v_k \leq S^{-N}(C)$  for every  $k$ ) or  $x \neq x$  otherwise. For the other cases, we will use induction on  $l, m$ :

- $l = 0$ : The resulting formula  $\exists x \left( \bigwedge_{j < m} S^N x < u_j \right)$  is equivalent to  $\bigwedge_{j < m} S^N(c) < u_j$ . Similarly when  $m = 0$ .
- $l = m = 1$ : In this case the formula has the form  $\exists x (t_0 < S^n x \wedge S^n x < u_0)$  which is equivalent to

$$S(t_0) < u_0 \wedge S^{n-1}c \leq t_0$$

- $k > 1$ : Note that the formula

$$\exists x \left( \bigwedge_{i < l} t_i < S^N x \wedge \bigwedge_{j < m} S^N x < u_j \right)$$

is equivalent to the formula

$$\begin{aligned} & \left( t_0 < t_1 \wedge \exists x \left( \bigwedge_{1 \leq i < l} t_i < S^N x \wedge \bigwedge_{j < m} S^N x < u_j \right) \right) \\ \vee & \left( \neg(t_0 < t_1) \wedge \exists x \left( t_0 < x \wedge \bigwedge_{2 \leq i < l} t_i < S^N x \wedge \bigwedge_{j < m} S^N x < u_j \right) \right) \end{aligned}$$

in which each of the formulas with quantifiers have less than  $l$  conjunctions, and by induction hypothesis are equivalent to quantifier free formulas. Note that the same argument can be used in the case when  $m > 1$ . ✓

**7.2. Uniform quantifier elimination for  $\mathcal{C}_P$  and quasi- $\mathcal{O}$ -minimality.** In this part we will show that the class  $\mathcal{C}_P$  defined in Section 3.2 has uniform quantifier elimination (in an expansion of the original language) and every infinite ultraproduct of  $\mathcal{C}_P$  is quasi- $\mathcal{O}$ -minimal.

First we show proof of uniform quantifier elimination.

Consider the extension  $\mathcal{L}^* = \{<, P, c, C, S, L, R\}$  of  $\mathcal{L}$  where  $c, C$  are constant symbols for the least and greatest element respectively,  $S$  is an unary function interpreted as the successor function, and  $L, R$  are 0-definable unary functions interpreted in the following way: for  $M \in \mathcal{C}_P$ ,  $M \models y = L(x)$  if and only if

$$\begin{aligned} M \models & (P(x) \wedge \neg P(y) \wedge \forall z (y < z < x \rightarrow P(z))) \\ & \vee (\neg P(x) \wedge P(y) \wedge \forall z (y < z < x \rightarrow \neg P(z))) \end{aligned}$$

Similarly for  $M \models y = R(x)$ . Basically,  $L(x)$  is the first change of sign for  $P$  where you go from  $x$  to the left. Similarly with the function  $R(x)$ .

Consider a primitive formula in the original language  $\mathcal{L}$ , say

$$\phi := \exists x \left( \bigwedge_{i < l} t_i < S^N x \wedge \bigwedge_{j < m} S^N x < u_j \wedge \bigwedge_{k < n} S^N x = v_k \wedge \bigwedge_{i,j,k} (P(t_i)^{p_i} \wedge P(u_j)^{p_j} \wedge P(v_k)^{p_k}) \wedge (P(x))^{p_x} \right)$$

where  $p_i, p_j, p_k, p_x \in \{0, 1\}$  in the notation  $\phi^0 := \phi, \phi^1 := \neg\phi$ .

We have to show that this formula is equivalent to a formula in which we just use the terms  $t_i, u_j, v_k$ . As in the previous section, if  $n > 0$  the formula  $\phi$  is equivalent to  $t_i = t_i$  or  $t_i \neq t_i$ , depending on whether the corresponding term  $x = S^{-N} v_k$  is a witness of the existential formula or not.

Also, the first two conjunctions will place  $x$  into a union of intervals, so we may assume that  $\phi$  has the form

$$\begin{aligned} \phi &:= \exists x \left( \left( \bigvee_{i < l} a_i < x < b_i \vee \bigvee_{j < n} x = c_j \right) \wedge \bigwedge_{i < l} (P(a_i)^{p_i^a} \wedge P(b_i)^{p_j^b} \wedge \bigwedge_{j < n} P(c_j)^{p_j^c}) \wedge (P(x))^{p_x} \right) \\ &\equiv \exists x \left( \left( \bigvee_{i < l} a_i < x < b_i \vee \bigvee_{j < n} x = c_j \right) \wedge (P(x))^{p_x} \right) \wedge \bigwedge_{i < l} (P(a_i)^{p_i^a} \wedge P(b_i)^{p_j^b} \wedge \bigwedge_{j < n} P(c_j)^{p_j^c}) \end{aligned}$$

So it is enough to find the equivalent formula corresponding to:

$$\begin{aligned}
\phi &:= \exists x \left( \left( \bigvee_{i < l} a_i < x < b_i \vee \bigvee_{j < n} x = c_j \right) \wedge P(x)^{p_x} \right) \\
&\equiv \exists x \left( \bigvee_{i < l} (a_i < x < b_i \wedge P(x)^{p_x}) \vee \bigvee_{j < n} (x = c_j \wedge P(x)^{p_x}) \right) \\
&\equiv \bigvee_{i < l} \exists x (a_i < x < b_i \wedge P(x)^{p_x}) \vee \bigvee_{j < n} \exists x (x = c_j \wedge P(x)^{p_x})
\end{aligned}$$

Again, every formula  $\exists x(x = c_j \wedge P(x)^{p_x})$  is equivalent to either  $c_j = c_j$  or  $c_j \neq c_j$ . On the other hand the formula  $\exists x(a < x < b \wedge P(x))$  is equivalent to either  $a \neq a$  (if  $S(a) \geq b$ ) or

$$\hat{\phi} := (\neg P(a) \wedge R(a) < b) \vee (P(a) \wedge R(a) \neq S(a)) \vee (P(a) \wedge R(a) = S(a) \wedge R(S(a)) < b).$$

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